

CHAPTER 6

LOGIC

One of the principal motivations for Bolzano's lifelong interest in logic was practical. The ability to think clearly and judge well, and to present useful knowledge in the most effective way, was a necessary condition for any durable political reform, and the study of logic would help to develop such skills. It was for this reason that Bolzano's lectures on religion at the Charles University contained such a wealth of logical instruction.

Another, equally important impetus for Bolzano's work came from his interest in the foundations of mathematics. The mathematics of Bolzano's day was quite different from that of today. Outside of elementary geometry and number theory, where the ancients had provided excellent models, proofs, where they existed at all, were often none too solid. The development of algebra, of analytic geometry, and the infinitesimal calculus had been rapid, but not always rigorous. Towards the end of the eighteenth century, increasing numbers of mathematicians started to look not only for new results but also for more solid foundations for what had already been discovered, seeking to introduce the rigour of the ancient geometers into the fields of algebra and analysis. Their work was continued in the first part of the nineteenth century by Abel, Cauchy, Dirichlet, Gauss, and others, whose work gave rise to a new and radically different understanding of mathematics. Bolzano, who while a professor of religion kept up his mathematical research in his spare time, was also among those looking into the foundations of mathematics at this time. Almost alone, however, he drew the appropriate philosophical morals from the new mathematics (which was partly of his own making), linking the reform of the foundations of mathematics with the reform of logic. For in his view, logic and mathematical method were one and the same, and as great and admirable as were the achievements of Euclid, Apollonius, and Archimedes, they had not said the last word on proof or axiomatics. This he held to be true not only with reference to branches of mathematics such as algebra or analysis, where the ancients had left little or nothing in the way of models, but also, somewhat surprisingly, in geometry itself.

In maintaining the centrality of logic in both philosophy and science, Bolzano was paddling against some strong currents of opinion. Among the philosophers of the modern period, it had become fashionable to denigrate

logic. Descartes thought he could sum up the entire discipline in four precepts: don't accept anything unless you are sure it is true, break large problems into smaller ones of manageable size, think in an orderly fashion, beginning with the simplest matters before moving on to the more complex, and when you make lists, don't leave anything out.¹ For his part, Hume, after stating a few rules for reasoning about causes and effects, declared:

Here is all the logic I think proper to employ in my reasoning; and perhaps even this was not necessary, but might have been supplied by the natural principles of our understanding. Our scholastic headpieces shew no such superiority above the mere vulgar in their reason and ability, as to give us any inclination to imitate them in delivering a long system of rules and precepts to direct our judgment in philosophy.²

Such celebration of natural reason, uncorrupted by instruction, was commonplace. Diderot's article on the syllogism for the *Encyclopédie*, for example, much of it cribbed from Locke's *Essay*, makes a few disparaging remarks on "artificial" (i.e., Aristotelian/scholastic) logic, and heaps praise upon what he calls "natural" logic. Natural logic is just the correct use of reason: to attempt to formalize it often hinders rather than helps.³ The thought that one should study logic is rejected as profoundly misguided: it would be like saying that we have to study anatomy and physiology in order to learn how to walk.

Apart from a handful of eccentrics like Leibniz, few seriously entertained the thought that any sort of logic, traditional or reformed, had a substantial role to play in philosophy or science. Christian Wolff, it is true, had tried to follow Leibniz in making logic central to philosophy, but his lack of skill probably did more to confirm the already low opinion most held of the subject.⁴ And Kant, who was at one time on the side of the logic-bashers, in the end made peace with it, finding a place for the traditional logic within

¹ *Discourse on Method*, Part Two.

² *Treatise*, 1.3.15, p. 175.

³ Diderot, Art. "Syllogisme," *Encyclopédie, ou Dictionnaire raisonné des sciences, des arts, et des métiers* (Berne and Lausanne, 1780).

⁴ Interestingly, Wolff impressed Frederick the Great sufficiently that this prince ordered that the drill of his officers include logical exercises, following Wolff's treatise (H. Arndt, editor's introduction to Ch. Wolff, *Vernünfftige Gedanken von den Kräften des menschlichen Verstandes* usw [Hildesheim: Olms, 1965], p. 96). Perhaps the memory of this practice partly explains Goethe's equation of logic with overly regimented, plodding, brutal thinking, even torture (*Faust* 1.1.4, tr. Kaufmann): "At first *collegium logicum* / There will your mind be duly braced / And well in Span-

his metaphysical system.¹ At the same time, however, he stated that no work remained to be done in logic, since Aristotle had already emptied the mine:

We have no one who has exceeded Aristotle or enlarged his pure logic (which is itself fundamentally impossible) just as no mathematician has exceeded Euclid.²

Moreover, the usual sort of logic had according to Kant only a minor role to play in philosophy (a new, “transcendental” logic that bore little relation to logic as usually conceived was required), and certainly had nothing at all to do with mathematics, which enjoyed its own method (the construction of concepts in intuition).

In part, the contempt of some moderns for logic came about for the same reason as the widespread contempt for religion: much of what was *called* logic was indeed contemptible. But as something that may reasonably be called logic is and always has been important, it would be surprising if it were utterly neglected. Descartes, whose views on logic we saw above, at the same time saw the central importance of a science he called *universal mathematics*, the general science of order and measure, which has more than a little in common with modern logic. More often, though, logical questions were raised and dealt with by practising scientists, who probably did not think to call what they were doing logic, and who rarely felt the need to expound the principles they employed in general works on logic or methodology.

Leibniz, of course, not only saw the importance of the the study of scientific and mathematical method, but also saw its connection with traditional logic. With him, too, begins the modern cross-fertilization of mathematics and logic—though the fact that he published so little meant that most of his work remained unknown in Bolzano’s lifetime. Bolzano, more impressed by the little he knew of his work than by that of Kant and his successors, made Leibniz’s project his own, and brought many parts of it to fruition. A rare

ish boots enlaced / So that more slowly than before / Thought creeps to execute its chore.”

¹ According to Herder’s notes [Ak 24, pp. 4–5], Kant had told his students in 1762/3 that Aristotle’s logic “had done the greatest damage” and that “to study scholastic logic is torture.”

² From the so-called Dohna-Wundlacken Logic, Kant, *Lectures on Logic* J. M. Young, ed. and tr. (Cambridge: Cambridge University Press, 1992), 438 [Ak 24, p. 700]. Cf. *Critique of Pure Reason*, B viii: “...since Aristotle, [logic] has not needed to retrace a single step.... Another remarkable fact about logic is that thus far it also has not been able to advance a single step, and hence is to all appearances closed and completed.” See also A xiv.

thinker in that he combined real mathematical and philosophical talent, Bolzano saw that mathematics and logic had much to learn from one another. Mathematicians needed logic because, surprising as it may sound, many of them didn't have a very clear idea of what proof amounted to. But logic also had much to learn from mathematics. If, as Leibniz had remarked, Aristotle had been the first (in his logical work) to think mathematically outside of mathematics,¹ in many respects he had barely scratched the surface, and not nearly enough had been done since to build upon his achievements. Many new discoveries in mathematics—notably the theory of functions—could also be put to use in logic.

All of this Bolzano saw. He understood, too, that while mathematics was an ideal test-bed and an important application for his logical innovations, the science of logic covered all the sciences, and especially philosophy. In his works, logic was placed at the very centre of philosophical research.

1. BOLZANO'S LOGICAL WRITINGS

Apart from a volume of sermons from 1813, Bolzano's earliest published writings deal with the foundations of mathematics.² A memoir of 1804, based on his doctoral work, deals with the foundations of geometry.³ In 1810, he published the first volume of his *Contributions to a Better-Grounded presentation of mathematics*,⁴ where among other things we find a classification of mathematical disciplines and a short presentation of logic. Three papers of 1816 and 1817 deal with the foundations of the calculus and geometry.⁵ All

¹ Letter to Gabriel Wagner (1696) in L. Loemker, ed., *G. W. Leibniz: Philosophical Papers and Letters*, 2nd edn (Dordrecht: Reidel, 1989), p. 465; cf. *New Essays*, IV, ii, §13.

² English translations of all of these early works may be found in *The Mathematical Works of Bernard Bolzano*, tr. S. B. Russ (Oxford: Oxford University Press, 2004). Hereafter, **MW**.

³ Bernard Bolzano, *Betrachtungen über einige Gegenstände der Elementargeometrie* (Prague, 1804).

⁴ *Beyträge zu einer begründeteren Darstellung der Mathematik*, erste Lieferung (Prague, 1810). Bolzano worked on a second instalment (See **BBGA**, Series 2A, Vol. 5), but decided not to publish it due to lack of interest in the first. Around the same time, he also wrote a short work entitled "Etwas aus der Logik" [**BBGA** 2A.5, pp. 139–68], which was later published in a Czech translation by O. Šír in 1831: "O logice," *Krok* 2 (1831) 55–78.

⁵ *Der binomische Lehrsatz ...* (Prague, 1816); *Rein analytischer Beweis ...* (Prague, 1817); *Die drey Probleme der Rectification, der Complination und der Cubirung ...* (Leipzig, 1817).

of these works, not just the *Contributions*, contain important reflections on methodological and logical questions.

Shortly after publishing the *Contributions*, with its brief presentation of logic, Bolzano decided to write a much more substantial work on the subject. His dismissal in 1819 finally gave him the opportunity to do so. Removed from the public sphere, he worked on logic throughout the decade of the 1820s, completing his greatest work, the *Theory of Science*, around 1830. It took several years to find a publisher—as we saw, publication within Austria was out of the question, even for a work on theoretical philosophy. In the end, Seidel in Sulzbach published it in 1837. During this long interval, Bolzano began work on an immense treatise of mathematics, which he planned to call the *Theory of Quantities* (*Größenlehre*).¹ Part of the introductory matter of the treatise is a brief presentation of the logic of the *Theory of Science* written for mathematicians, called “On the Mathematical Method.”² It was actually this shorter version of Bolzano’s logic (which first appeared in print in the 1970s) that was first read and discussed. In 1833, Bolzano sent a copy of the essay on Mathematical Method to Franz Exner, the newly appointed professor of philosophy at the Charles University. The two then discussed aspects of Bolzano’s logic, in person and in a series of letters, most intensely during the years 1834 and 1835.³ Bolzano profited from these exchanges, making a number of changes to the *Theory of Science* before it was finally published. Other corrections, too late to be incorporated in the published version of the work, are preserved in a manuscript entitled “Emendations and additions to the Logic,” which has been published in the *Bolzano Gesamtausgabe*.⁴ The late works *Dr. Bolzano and his Opponents* (1839) and *A Critical Survey of Bolzano’s Theory of Science and Science of Religion* (1841), finally, are valuable sources for Bolzano’s logical views.⁵

2. THE SCOPE OF BOLZANO’S LOGIC

For Bolzano, logic has broader scope than most philosophers give it today. It is a *theory of science*, which treats of everything having to do with the

¹ **BBGA**, Series 2A, Vols 7–10; some parts have yet to be published.

² =**ML** [**BBGA** 2A.7, pp. 46–97; Eng. tr. in **MM-EX**, pp. 40–82].

³ The relevant letters are translated in **MM-EX**.

⁴ “Verbesserungen und Zusätze zur Logik,” **BBGA** Series 2A, Vol. 12/2. This volume also contains some other manuscripts and letters bearing on Bolzano’s logic.

⁵ *Dr. Bolzano und seine Gegner* (Sulzbach, 1839; new edn, **BBGA** Series 1, Vol. 16/1); *Bolzano’s Wissenschaftslehre und Religionswissenschaft in einer beurtheilenden Uebersicht* (Sulzbach, 1841; new edn planned for **BBGA** as Series 1, Vol. 17).

organization and presentation of knowledge. It is, first of all, a theory of scientific composition, which instructs us on how properly to divide human knowledge into distinct sciences and present these sciences in appropriately structured treatises (Book 5 of the *Theory of Science*). With typical thoroughness, Bolzano demands that all the layers supporting a finished science also be explored. A completed science presupposes a science still under construction, which motivates his inclusion of *Heuristics*, or *The Art of Discovery*, within logic (Book 4). The discovery of truth in turn presupposes the ability to form concepts and judgments, to make inferences, and so on, which are the topics of the *Theory of Knowledge* (Book 3). But the decisive step is taken when Bolzano goes beyond this layer of knowledge or subjective activity upon which Kant, along with most of the moderns, wanted to found objective or scientific knowledge. For Bolzano, the foundations of a science are to be sought in the *meaning* of its claims, in the abstract, atemporal element that they contain. The collection of subjective ideas and judgments is thus paralleled by the collection of objective, logical entities that correspond to them, which Bolzano calls *ideas and propositions in themselves*. The “existence” of these ideal entities (existence in the mathematical sense, for logical objects are not real according to Bolzano) is proven in the *Theory of Fundamentals* (Book 1); the theory of propositions and ideas in themselves constitutes logic in the narrow, contemporary sense of the term. The *Theory of Elements* (Book 2), which is devoted to these matters, is the most original and important part of the *Theory of Science*. Bolzano’s logic is a formal logic, which, however, does not study the formal aspect of actual thought, but rather, in a completely new departure, concerns itself with the forms of propositions in themselves.

In many ways, Bolzano remains close to and respects the logical tradition. The basic plan of his theory of science does not differ much from Aristotle’s. Like Aristotle, he speaks of the division of sciences, the deductive order within individual sciences, of the nature of demonstration, of the analysis of propositions, and of the nature of ideas, or terms. The impression of continuity and familiarity is easily confirmed by a superficial reading of the *Theory of Science*. For Bolzano had a habit of retaining accepted terminology and ways of speaking wherever possible. Because of this, he may sometimes seem merely to be repeating what others have said before him. This is, in fact, quite rare with him: Bolzano’s logic represents one of the greatest revolutions in the history of the subject, something that is masked to a great extent by the adoption of customary ways of speaking. A talented mathematician, he had a skill that not many philosophical authors (and readers) possess: the ability to clearly define his terms, and to adhere faithfully to the meanings set

out in the definitions. Few writers have maintained this level of clarity and precision while still being misunderstood. The key lies in retaining his definitions in one's memory, and not substituting other meanings, drawn from other philosophers or ordinary usage, in their places.

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In the present chapter, we will speak primarily of Bolzano's logic in the modern sense, that is, we shall focus on the *Theory of Elements*. Apart from a few remarks on the epistemology of logic (Section 7d) that seem to belong here, we postpone discussion of Bolzano's *Theory of Knowledge* until Chapter 7. We shall not say much at all in this book about Bolzano's *Art of Discovery*,¹ and shall only briefly discuss the general account of scientific method expounded in the *Theory of Science Proper*, though we shall speak later of Bolzano's views on *mathematical* methodology (Chapter 9). Finally, though we shall discuss Bolzano's concept of deducibility and related notions in detail, we shall not touch upon the Theory of Inferences" (Part Four of the *Theory of Elements*).

3. LOGICAL OBJECTS

Contemporary logicians, taking for granted the methods of modern mathematics, suppose without further ado a variety of abstract objects of inquiry. "Languages" are postulated containing infinitely many symbols (for predicates, individual constants, variables, etc.), which may be combined in various ways, some of these combinations resulting in well-formed formulas, sequences (e.g., proofs) or other collections (theories, etc.) of formulas, and so on. These are not the languages of everyday speech, but rather abstract structures that sometimes connect more or less well with parts of ordinary or scientific language. It is rare to find a logician who thinks it worthwhile, still less necessary, to justify speaking of such abstract objects in logic, which has become a matter of routine.

Here, for instance, is what a few contemporary logicians propose:

We assume we have available an infinite list of *one place relation symbols*, $P_1^1, P_2^1, P_3^1, \dots$, an infinite list of *two place relation symbols*, $P_1^2, P_2^2, P_3^2, \dots$, and so on. [...]

¹ For a discussion of the *Art of Discovery*, see the editors' introduction to Volume 3 of the *Theory of Science*.

We also assume we have available an infinite list of *variables*, $v_1, v_2, v_3, \dots$.¹

We assume we are given an infinite sequence of distinct objects which we will call symbols. . . .²

A formal theory $\mathcal{T}$ is defined when the following conditions are satisfied:

1. A countable set of symbols is given as the symbols of $\mathcal{T}$. A finite sequence of symbols is called an *expression* of $\mathcal{T}$. Etc.³

Things were different in Bolzano's time. While he had a conception of logic that does not greatly differ from those common today, he had no abstract mathematics to appeal to, for it did not yet exist. Writing for an audience of philosophers with little logical sophistication and often no knowledge of mathematics, many of whom looked upon logic as a branch of psychology, the art of thinking, the science of correct thought, or what have you, he could by no means assume that his readers would allow him the use of abstract entities or even understand him when he appealed to them in his logical theory. It is at least partly for this reason that we find in his works a detailed explanation and defence of the use of abstract entities in logic (a fair proportion of Bolzano's correspondence with Exner is also devoted to the topic).

There are many ways to show the need, or at least the usefulness, of speaking of such entities. In propositional logic, for example, one might wish to prove the following theorem:

The argument form 'Not both A and B , $A \therefore$ Not- B ' is valid.

That this is so is usually shown by considering the various possibilities (A and B could both be replaced by truths, one could be replaced by something true and the other by something false, or both by something false) and showing that under none of these circumstances would the premises of the argument turn out to be true and the conclusion false. We depict this state of affairs with the help of a truth table:

¹ M. Fitting and R. L. Mendelsohn, *First-Order Modal Logic* (Dordrecht: Kluwer, 1998), p. 81.

² H. Enderton, *A Mathematical Introduction to Logic* (New York: Academic Press, 1972), p. 17.

³ Elliott Mendelson, *Introduction to Mathematical Logic* 4th edn (Boca Raton: CRC Press, 2001), p. 34.

<i>A</i>	<i>B</i>	Not both <i>A</i> and <i>B</i>	<i>A</i>	Not- <i>B</i>
<i>T</i>	<i>T</i>	<i>F</i>	<i>T</i>	<i>F</i>
<i>T</i>	<i>F</i>	<i>T</i>	<i>T</i>	<i>T</i>
<i>F</i>	<i>T</i>	<i>T</i>	<i>F</i>	<i>F</i>
<i>F</i>	<i>F</i>	<i>T</i>	<i>F</i>	<i>T</i>

Let us now ask ourselves: what do we need to know about the items that can take the places marked by *A* and *B*? Judging from the above table, and the proof that accompanies it, simply that they are something that is either true or false, but not both. For this part of logic, at least, little more seems to be required.¹ So to state the theorems of this part of logic, it is sufficient to speak of entities that satisfy these conditions.

The usual treatments of logic, Bolzano observes, state theorems similar to the one discussed above only for *judgments* made by some thinking being. But the condition that a proposition is thought, asserted, or judged to be true clearly plays no role in the above proof. We have no need to speak of thoughts, linguistic expressions, or indeed of anything that exists in the world in order correctly to formulate or prove these theorems. Dropping these conditions, we arrive at the concept of a proposition in itself, which he describes in the *Theory of Science* as “any statement that something is or is not, regardless of whether it is true or false, whether or not somebody has put it into words, and even whether or not it has been thought.”² In a letter to Franz Exner, he writes that he hopes people will understand what he means by ‘proposition in itself’ when he tells them that a proposition in itself is either true or false (but not both), but does not have actual existence.³ In modern terms, a proposition in itself is an abstract object with a truth-value.

Stating the theorems of logic for propositions in themselves, rather than for propositions that have been thought, judged to be true, or expressed, gives them the greatest possible generality. Sound methodology thus argues in favour of referring to propositions in themselves in logic:

¹ Careful readers will notice that we also need to suppose that these entities can enter into combinations with the operators “Not both ... and ...” and “Not ...”, that wherever *A* occurs *B* could also occur, and a few other things besides.

² **WL**, §19 [I.77].

³ Bolzano to Exner, 22 November 1834 [**BBGA** 3.4/1, pp. 82–3; **MM-EX**, p. 141]. Propositions with empty, or objectless, subject-ideas are deemed false by Bolzano, as are those with empty predicate-ideas. See below, p. 252.

It is obvious that most of the theorems set out in the section entitled “on judgments” hold good not only of judgments but of propositions in general, regardless of whether they have been thought or asserted. As it is a rule of good exposition not to describe something as holding under certain conditions if it is unconditionally valid, one should for this reason alone introduce the aforementioned concept [*sc.*, of a proposition in itself] into logic.¹

Bolzano also takes pains to show that ordinary linguistic usage refers to, and quantifies over, propositions in themselves, and thus accepts that there are such things.² To see that this is so, one can ask whether there are any truths that are completely unknown to mankind. It seems likely, for instance, that no one in ancient times knew that fibre-optic cables could be used to carry signals, and just as likely that the thought never even occurred to anyone at that time, since almost certainly no one had even formed the required concepts. Doubtless there are similar truths the existence of which is unsuspected today. Perhaps there are truths that no one shall ever know, that shall never be expressed in any language, and that no one shall ever even consider. Now many people will admit that what we have just said makes perfectly good sense, and is indeed true. But what is it, exactly, that we were speaking about, and indeed quantifying over? What is a truth that no human being is aware of, that no one has ever considered? It seems clear that it cannot be a human thought, since we assumed that no one has thought or ever will think it. Perhaps truth is in things: a truth might just be a way things are. What then to make of a falsehood? Is it supposed to be the way things aren’t? That doesn’t sound right somehow. But if falsity isn’t in things, then it seems reasonable to suppose that truth isn’t either. There must, therefore, be a third kind of thing, what Bolzano calls a *truth in itself* or a *truth as such*, and more generally *propositions in themselves*, which can be either true or false.³

Another way to understand Bolzano’s point is to try to *count* propositions. If propositions were just thoughts of a certain kind, then to count them we would need to count thoughts. But it is clear first of all that by merely

¹ WL, §20.1 [I.81].

² See, e.g., Bolzano’s letter to Exner of 22 November 1834 [BBGA 3.4/1, pp. 86–7; MM-EX, pp. 144–5.]

³ Here we reproduce the argumentation from Leibniz’s “Dialogue on the connection between things and words,” pp. 182–5 in *Philosophical Papers and letters*, ed. L. Loemker, 2nd edn (Dordrecht: Kluwer, 1989). Included in Raspe’s edition, this essay was well known to Bolzano.

counting certain thoughts we will miss a great many propositions (those that haven't yet been or never shall be thought), and at the same time that we shall count many of them more than once. Probably over five billion people alive today have thought, for instance, that the sky is blue. Are there then that many distinct propositions? If I think that the sky is blue several times, do I thereby multiply the number of propositions? It seems more natural to say that there is *one* proposition that is thought by different people, or on different occasions by the same person. Similar remarks hold for linguistic expressions, be they written or spoken. There may be many expressions (types or tokens) for a given proposition (e.g., 'The sky is blue', 'The sky is blue', '*Le ciel est bleu*', '*Nebe je modré*', '*Der Himmel ist blau*', etc.), but also propositions for which no expression exists. The things spoken of here, the things that are true (or false), whatever they are, do not appear to be thoughts, nor written or spoken expressions, but something else altogether.

In response to a review of the *Theory of Science*, Bolzano tells us that these are not eccentric claims. Rather, to judge from ordinary usage, they are widely accepted:

[A]s proof that we are not alone in seeing things this way, we call upon the common human understanding, which divides truths into known and *unknown*, and has no objection to admitting that there are also truths which *no one* thinks of (with the possible exception of the omniscient God). None of this could occur if truths were taken to be nothing other than a kind of *thoughts*. Thousands of people will also grant the following claim without hesitation: "If there were no thinking beings, it would be *true* that there were no thinking beings." [...] Finally, we bid the reviewer to think an arbitrarily chosen proposition, e.g., $2 \times 2 = 4$, along with us. Beyond a doubt, *his* thought along with *ours* will together amount to *two thoughts*. Will he also claim that *two truths* are thought here? Certainly not, rather only a single truth. This one item, then, which B. calls a truth *in itself*, is obviously completely different from a thought, and in general is not something that exists.¹

Following Aristotle in thinking that good ideas are unlikely to be original, Bolzano finds support for his position in the fact that a number of other

¹ Dr. B. Bolzano und seine Gegner (Sulzbach, 1839), p. 150 [BBGA 1.16/1, p. 129]. Bolzano writes in the third person in this work in order to conceal his authorship, as he was still forbidden to publish.

philosophers have employed similar concepts. The Stoic sayables (*λεκτα*) are perhaps the closest ancient approximation to his propositions. He only refers to the Stoics at second hand, however, via a reference to Sextus Empiricus, who says that according to them, *axioms* are that which is true or false.¹ Leibniz's notion of a possible thought (*cogitatio possibilis*) is recognized as a clear anticipation of his notion,² though he adds later that the expression is misleading, because "a possible thought is not a kind of thought, but merely a kind of possibility."³ Contemporaries such as Mehmel and Herbart are also recognized as embracing similar conceptions. In a letter to Exner, he quotes freely without indicating his source, probably a neoscholastic textbook: "You are aware that I have a considerable number of philosophers on my side (those who teach of so-called *veritatem metaphysicam objectivam, qua nemine cogitante rem edicit, uti se habet* [metaphysical, objective truths, which, even if no one thinks them, state something of a thing that the thing has])."⁴ At another place, a note of his mentions the link to the Scotists.⁵

Propositions in themselves are just one of several kinds of abstract object Bolzano appeals to in his logic. There are also *collections* of propositions, some of them constituting *sciences*. In their turn, propositions have parts, which (provided they are not complete propositions) Bolzano calls *ideas in themselves*.⁶ Ideas in themselves stand in certain relations to each other, as do propositions in themselves, and they do so completely independently of anyone's thought. There are, for instance, infinitely many pairs of numbers *x* and *y* such that *x* is a multiple of *y*. For any one of these pairs, any object standing under the idea [multiple of *x*] will also stand under the idea [multiple of *y*], and would do so regardless of whether anyone has ever thought this.⁷ Similarly, the proposition [There were 14 Canadian Prime Ministers in

¹ WL, §23, no. 2.

² WL, §21, no. 3.

³ WL, §23, no. 1 [I.92].

⁴ Letter of 9 July 1833 [BBGA 3.4/1, p. 23; MM-EX, p. 92].

⁵ BBGA 2B.18/2, p. 45.

⁶ *Vorstellungen an sich*. 'Vorstellung', which we translate as 'idea', is also sometimes translated as 'presentation' or 'representation'.

⁷ It is common in the literature on Bolzano to place square brackets around a sentence to form a designation of the proposition in itself it expresses, and we shall follow this convention, applying it not only to sentences but also to other similar entities. Single quotes are used to indicate mention of linguistic expressions. For example, '[Socrates has wisdom]' will designate the proposition in itself expressed by 'Socrates has wisdom', while '[wisdom]' designates the idea in itself designated by 'wisdom'.

the twentieth century] entails the proposition [At least two twentieth-century Canadian Prime Ministers were born in the same month] independently of anyone thinking that this is so. In the first instance, logic must describe these relations between propositions and ideas in themselves, making no reference to the mind, its faculties or activities.

In addition to the abstract propositions in themselves, Bolzano also speaks of *thought* or *expressed* propositions (propositions in themselves are also sometimes called *objective* and thought propositions *subjective*). Objective propositions may also be thought of as the *matter* of the subjective, and as the *meanings* of certain sentences.¹ Similarly with ideas: the idea in itself can be considered the matter of a subjective idea, and as the meaning of certain expressions. One can say that when one thinks a subjective idea or proposition, one by the same token *grasps* the corresponding objective entity. One should not be misled, however, by the term *grasp*, which is of necessity purely figurative, there simply being no words that are not.² Not existing in space or time, propositions in themselves have no causal powers. Bolzano will say that *there are*³ propositions in themselves, but they do not actually exist, interpreting the first part of the assertion to mean that the *idea* [proposition in itself] has objects, or, more precisely, has the property of having objects, for which he coins the term ‘objectuality’ (*Gegenständlichkeit*). The concept of objectuality corresponds closely if not exactly to the existential quantifier of contemporary logic: it applies not to objects, but rather to ideas. Actual existence, or actuality, by contrast, *is* in his view a property of objects: trees have it, abstract objects such as propositions in themselves lack it.

* * *

Some later logicians, notably Quine, would reject propositions, preferring to speak only of *sentences*. Bolzano was never confronted with arguments like Quine’s, so it is difficult to guess how he might have reacted to them. Still, in view of the quality of these arguments and their influence, it seems worthwhile to say a few words about Quine’s position and its bearing on Bolzano’s views.

To begin with, it should be noted that what Quine calls sentences in the context of logical theory are complex set-theoretical constructions, every bit

¹ **ML**, §2 [**BBGA** 2A.7, p. 47; **MM-EX**, p. 40]. In some cases, e.g., sentences containing indexicals, a single sentence type could be used to express various propositions depending upon the context of utterance.

² Bolzano to Exner, 18 December 1834 [**BBGA** 3.4/1, p. 106; **MM-EX**, p. 164.]

³ In German, the verb ‘to be’ is not even used: *Es gibt Sätze*.

as abstract as Bolzano's propositions. Briefly, Quine suggests that we define phoneme-types as equivalence classes of actually uttered phonemes, and sentences as certain recursively specified sequences (in the mathematical sense) of such phoneme-types. Written sentences would be dealt with in a similar way. Sentences are thus types, of which utterances and inscriptions are the tokens.¹ Quine's objection to propositions was not that they are abstract objects, but rather that, if propositions were taken to be the meanings of sentences, he could see no effective way of individuating them and—a related point—no way of determining in every case which proposition was the meaning of a particular sentence.² Bolzano seems prepared to concede Quine's point on several occasions. For example, in the essay on mathematical method, he states that a proposed definition of a term should be deemed successful if it gets the extension right:

One might, for example, want to dispute forever about whether we have given the correct concept of the expression "extended spatial object" when we define it as a spatial object of such a kind "every one of whose points, at every distance no matter how small, has certain neighbours"—if we are in a position to derive from this concept all the properties that one knows of extended spatial objects, then it will be shown that our concept, if not identical with the customary one, is at least equivalent to it, and one will have cause to be satisfied with it.³

And in the *Theory of Science*, he makes a similar suggestion concerning the analysis of judgments, i.e., thought propositions:

If we have formed a proposition *M* with ideas $\alpha, \beta, \gamma, \dots$ that seems to be completely identical to the judgment *A* we are supposed to define, the correctness of this surmise will be confirmed mainly by our ability to deduce the same consequences from *M* as we can from *A*. Admittedly, this really only shows, strictly speaking, that the two are *equivalent*.⁴

¹ *Word and Object* (Cambridge: MIT Press, 1960), §40. *Philosophy of Logic*, 2nd edn (Cambridge: Harvard University Press, 1986), Chapters 1, 2; pp. 55–6.

² See, e.g., *Philosophy of Logic*, 2nd edn, p. 3.

³ **ML**, §11 [**BGA** 2A.7, pp. 71–2; **MM-EX**, p. 61]. Here, by *equivalent*, Bolzano means *co-extensive*. See below, p. 219. Cf. **WL**, §668.9 [IV.548].

⁴ **WL**, §366 [III.449–50]. Here, *equivalent* means *mutually deducible*. Cf. **WL**, §137 [II.53–4], quoted below, p. 244; see also below, p. 309.

This being said, Bolzano clearly thought that we could in some cases do better than this. He continues the above passage as follows:

But if the constituents of which *M* is composed are all found in *A* as well, and our most carefully scrutiny reveals no parts in *A* that are not also in *M*, nor any parts combined in different manners in the two, we may be allowed to surmise that we have correctly indicated the way *A* is composed in our definition.¹

Note, however, his continued caution—“we may be allowed to surmise”, not “we may be certain”. And Bolzano nowhere states that analysis is guaranteed to succeed in every case.

Lacking the technical apparatus of formal syntax, Bolzano did not have anything like Quine’s “sentences” at his disposal. Given his remarks above, we expect that he would have conceded that we are in no position to provide a definitive correlation of actual utterances with propositions. On the other hand, we expect that Quine would have agreed with Bolzano that it is not expedient to formulate the theorems of logic for actually formed utterances or inscriptions² and that something else—some sort of abstract object—is, if not absolutely indispensable, at least useful. It seems to us, too, that Bolzano’s propositions in themselves (or at least ersatz versions of them) would be found to exist in the vast set-theoretic universe that Quine seems on many occasions prepared to accept—the transfinite cactus in the desert landscape. The crucial questions, in our view, would bear on: (1) whether any given specification of a set of “sentences” could be thought to be exhaustive of all possible forms of meaningfulness; and (2) the relations between the abstract logical objects (propositions or “sentences”) and the everyday utterances and inscriptions produced by human beings.

Concerning the first point, accepting propositions makes room within logical theory for the discussion of the possible future extension of forms of meaning. Quine himself allows that future languages may contain devices that are unknown today.³ To this extent, anyway, he might have been prepared to concede to Bolzano the usefulness of speaking of propositions over and above sentences. The wisdom of proceeding in this way will, we hope, become evident later in this chapter, when we turn to Bolzano’s treatment of logical consequence and related notions. Precisely because he framed his

¹ **WL**, §366. For similar remarks with respect to ideas, see **WL**, §668, no. 9.

² See, e.g., *Methods of Logic*, 4th edn (Cambridge: Harvard University Press, 1982), p. 4; *Word and Object*, pp. 194–5.

³ *Philosophy of Logic*, p. 34.

definitions quite generally, in terms of propositions and their parts, not tying his account to a particular syntax, Bolzano was able to provide theories that can still be quite usefully applied today.

With respect to the second point, it is not obvious to us that the relation between the “sentences” studied in logic and sentences (i.e., actual utterances or inscriptions) is any less problematic than the relation between propositions and sentences. This point may be obscured by Quine’s use of ‘sentence’, which may lead one to think that he is in fact speaking of actual utterances, inscriptions, etc., rather than abstract objects. Allow us to call the latter *quentences* from now on, reserving ‘sentence’ for individual utterances or inscriptions.

Quentences are quite different from sentences. To begin with, there are infinitely many quentences, but given some eminently reasonable assumptions there can be only finitely many sentences. Quine seems occasionally to lose sight of this point. In his *Philosophy of Logic*, for example, he describes the grammarian’s task as follows:

The grammarian’s question is, then, what strings of phonemes belong to the language? What strings, that is, ever get uttered or *could get uttered* in the community as normal speech?¹

He is supposed to do this *formally*, i.e., via a mathematical theory of finite strings of phonemes. Quine continues:

[T]he desired strings, though finite in length, are infinite in number.²

Now, if we assume that there are only finitely many phonemes and that strings are finite in length, Quine’s claim can only be true if arbitrarily long strings could get uttered. If we further assume that only finitely many phonemes can be uttered in any given time, and that linguistic communities have only finitely many members, Quine’s claim would also presuppose immortality.

Later, to be sure, we are told that the demands of simplicity of theory sometimes trump empirical considerations:

[The grammarian’s] purpose is ... to demarcate formally, *in a reasonably simple and natural way*, a class of strings of phonemes which will include practically all observed utterances and exclude *as much as practicable* of what will never be heard.³

¹ *Philosophy of Logic*, 2nd edn, p. 16, emphasis added.

² *Philosophy of Logic*, 2nd edn, p. 16.

³ *Philosophy of Logic*, 2nd edn, p. 22, emphasis added.

If this is so, however, why not put an upper bound on the length of strings, since we can be quite confident, e.g., that sequences of $10^{10^{10}}$ phonemes will never get uttered by anyone in our linguistic community? Since Quine never proposes this (such a move would entail the loss of many important metalogical theorems, for one thing), we conclude that in his view logic is, in the first instance anyway, concerned more with quentences than with sentences.¹

Moreover, quentences are tidy in ways that ordinary sentences tend not to be. Bivalence governs quentences, extensions of the general terms occurring in them are sharply delimited as only classical logic can make them,² and ambiguity is, if not unheard of, at least thought to be unproblematically eliminable.³

The last point merits amplification. In an article of 2006, Wolfgang Künné, drawing on some observations of Strawson, has pointed out some of the problems confronting Quine on this score.⁴ Consider, for example, the sentence:

- (1) If plucking geese gets you down, then plucking geese gets you down.

Is this sentence a logical truth? Syntactically, it is of the form ‘If P , then P ’, which Quine recognizes as a valid schema, yet, depending upon how one understands the antecedent and consequent, it may well not be true, still less logically true.⁵

¹ Note, too, that even if we correlated utterances with quentences purely syntactically, the number of quentences concerning which we have behavioural data would be infinitesimally small in comparison with the number of those for which we have none.

² See W. V. O. Quine, “What price bivalence?” *Journal of Philosophy* **78** (1981) 90–5.

³ See, e.g., *Methods of Logic*, 4th edn, pp. 4, 56–7.

⁴ “Analyticity and logical truth; from Bolzano to Quine,” pp. 184–249 in M. Textor, ed., *The Austrian Contribution to Analytic Philosophy* (London: Routledge, 2006), pp. 228 ff.

⁵ The problems adverted to here, having to do with so-called “token-synonymy”, were brought to Quine’s attention by P. Strawson (“Propositions, concepts, and logical truth,” [1957] in P. Strawson, *Logico-linguistic Papers* (London: Methuen, 1971), pp. 116–29). In a brief response (“Reply to Strawson”, [1969] in D. Davidson and J. Hintikka, eds, *Words and Objections* [Dordrecht: Reidel, 1975], pp. 320–5), Quine conceded Strawson’s point, and abandoned his earlier definition of logical truth. The kluge he suggests, however, while it perhaps allows him to continue to speak of logical truth, is still open to serious objections. See Künné, “Analyticity and logical truth,” pp. 228ff. for details of some of these.

The difficulty extends far beyond cases like that cited above, where we all recognize the presence of ambiguity. The same problem arises even in cases like the following:

2 is prime or it is not the case that 2 is prime.

This may well be counted as true by most speakers of English, but to count it as *logically* true, Strawson argues, seems to require more than this—in particular, it seems to require the assumption that the two occurrences of the sentence-type ‘2 is prime’ are *synonymous*, or *necessarily equivalent* or something of the sort.¹

But if logical truth is defined for quentences (where no such ambiguities occur), it seems that this property may or may not be inherited by actual sentences that are tokens of a given quentence (alternatively, we might say that all tokens of a logically true quentence are logically true, but it remains uncertain in general whether a given sentence is indeed a token of a given quentence). It then becomes something of a mystery how we can learn about the logical properties of sentences by studying quentences. And while one might attempt to forge a link between quentences and sentences by fiat, maintaining that logical theory only applies to a certain well-behaved subset of actual sentences, it seems to us that it is not altogether obvious that we are in a position to determine (by behaviour or other empirical means) just what this subset is, or indeed if it is non-empty.²

Moreover, given the plurality of logical systems, and disputes even among the learned concerning the validity of principles as basic as bivalence, excluded middle, or non-contradiction, Quine’s attachment to classical first-order logic seems impossible to justify on empirical grounds. The claim that the truths of logic (by which Quine means classical first-order logic) are all *obvious* (where obviousness is to be fleshed out in behavioural terms)³ is, as Graham Priest has observed, “mind-numbingly false”,⁴ and talk of *deviant*

¹ Cf. Strawson, “Propositions, concepts, and logical truth.” Merely having the same truth-value seems insufficient, since this would allow us to count an instance of (1) as a logical truth even if we understood the antecedent and consequent differently, provided that they had the same truth-value.

² Similarly, in *Philosophy of Logic*, Quine tells us that even when logic considers so-called “eternal sentences”, it does so on the understanding that it considers them relative to a language community and a time (p. 14). It is far from obvious that languages can be individuated in the required way by empirical means.

³ See, e.g., “Carnap and Logical Truth,” in *Ways of Paradox*, rev. edn (Cambridge: Harvard University Press, 1976), p. 111; *Philosophy of Logic*, 2nd edn, pp. 82–3.

⁴ *Doubt Truth to be a Liar* (Oxford: Oxford University Press, 2006), p. 172.

logics (and, correlatively, of *our* logic) mere exhortation.¹ Once again, it seems it is the desire for simplicity of theory, rather than empirical data, that is driving the bus.²

Perhaps it is better to look upon logic as dealing with idealizations (of language if you wish, though some would, with Bolzano, say also of discursive or propositional *thought*), constructing models that illuminate some features of language while perhaps inevitably doing violence to others, and not allowing of any fully determinate mapping of actual utterances or inscriptions onto the set of objects studied by logic. In this respect, propositions would seem to be in much the same situation as quentences.

* * *

By formulating his fundamental logical theories for propositions and ideas in themselves, Bolzano spotted, and avoided, the besetting sin of the logic of his time, namely, *psychologism*, the view that logic studies the laws of thought, the operations of the mind, and so on. Later, both Frege and Husserl engaged in fierce polemics against psychologism in logic. Many of their arguments—to the effect that a descriptive account of how the mind works cannot validly give rise to normative laws of logical thinking, that taking a proposition to be true is not the same thing as the proposition being true, that psychologism has a strong, perhaps irresistible tendency to collapse into subjectivism and relativism, and hence into triviality or self-refutation—may all be found in various places in Bolzano's *Theory of Science*.³

¹ *Philosophy of Logic*, Chapter 6.

² Cf. *Word and Object*, §47 (p. 227): "Laws of logical inference refer to recurrences of sentences, on the assumption that a sentence true in one occurrence will be true in the next. Even inference of '*p*' from '*p* and *q*' (where '*p*' and '*q*' represent sentences) is a case in point. Any plan not predicated thus on fixity of truth values would be unrewardingly complex."

³ Husserl was in fact accused of having pilfered his arguments from Bolzano by M. Palagyi in his book *Der Streit der psychologischen und Formalisten in der modernen Logik* (Leipzig, 1902). Husserl replied to the charges in a review of 1903 (Eng. tr., "A reply to a critic of my refutation of logical psychologism," in J. Mohanty, *Readings on Edmund Husserl's Logical Investigations* [The Hague: Martinus Nijhoff, 1977]). This accusation was quite unfair, as Husserl had generously acknowledged Bolzano's influence in the *Logical Investigations* (*Logische Untersuchungen* Vol. 1. *Husserliana* Vol. 17 [The Hague: Martinus Nijhoff, 1975], pp. 227 f.). To this point, no hard evidence has been found which proves that Frege read Bolzano's works. Künne ("Propositions in Bolzano and Frege," *Grazer phil. St.* 53 (1997) 203–40, p. 221) argues quite convincingly that, at least at the time of writing

Given this, some readers may be surprised when Bolzano announces that logic is not an *independent* science, and in fact depends precisely upon the science of psychology.¹ This is no more than a terminological issue, however, stemming from Bolzano's broad conception of his subject, which we mentioned above. After the development of the abstract foundational part in the first two volumes of the *Theory of Science*, he turns in the third volume to epistemological matters, discussing the problems of how truths may be discovered, error avoided, and so forth. At this point, psychology begins to play a role. In the properly logical part, the theory of concepts, propositions, and arguments, things stand otherwise. He did not, as we shall see, make judgments and inferences, mental occurrences, the foundation of logic, but propositions in themselves, ideas in themselves, and the relations among them. In *this* part of his logic (which is more or less the whole of logic from the contemporary perspective), psychology has no place.

4. PROPOSITIONS AND IDEAS

Compared to previous treatments of the subject, Bolzano's exposition of logic proper begins with a small change of detail, to all appearances a minor one. Since the time of Aristotle, it had been customary to begin with a discussion of terms (corresponding to Bolzano's ideas), and then move on to propositions (or judgments), simple arguments (or syllogisms), chains of arguments (or demonstrations), and finally to sciences (ordered collections of propositions along with their proofs). A reasonable order of exposition, it was based upon the observation that terms were the elements of propositions, propositions the elements of arguments, and propositions along with their supporting arguments the elements of sciences. To proceed in this way was thus to move from the simple towards the complex.

Bolzano reversed the order of the first two items, dealing first with propositions and only afterwards with ideas. He did this because he saw that the traditional approach had not been worked out in sufficient detail and that prospects were not good for filling in what was lacking. The usual treatments identified only a handful of elements of basic propositions, usually

the *Foundations of Arithmetic* (published 1884), Frege had not read the *Theory of Science*; Sundholm ("When, and Why, did Frege read Bolzano?" *The Logica Yearbook* [Prague: Filosofia, 1999], pp. 164–74) argues forcefully for the claim that Frege did read Bolzano, but probably considerably later, around 1904. Künne (*Versuche über Bolzano* [St Augustin: Academia, 2008], pp. 330 ff.) is more cautious.

¹ WL, §13 [I.54].

singular terms (e.g., [Socrates]), general terms (e.g., [human]), the copula ([is]), negation, and quantifiers (e.g., [all], [some]). Basic propositions were then described as certain combinations of such elements, e.g., [Socrates is human], [All men are mortal], [No man is omniscient], which in turn could be combined to form hypothetical propositions (e.g., [If all men are mortal then Socrates is mortal]) disjunctive propositions (e.g., [Either all men are mortal or some men are not mortal]) and the like; finally, *modal* propositions, such as [Necessarily, no square number is prime] might be formed. Basic propositions, it was claimed, were all of the subject–predicate form; each had its quantity (universal, particular, or singular) and its quality (affirmative or negative).

Clearly there was agreement on some of the features of propositions. But had anyone actually given a viable definition of the concept of a proposition? A proposition was said to be a certain kind of combination of terms (or concepts, ideas, etc.). But not all combinations of terms form propositions, for example: [All not some], [Socrates Plato or]. And no one had so far produced a non-circular definition that determined which combinations were and which were not propositions.¹

What is more, the usual theories greatly oversimplified their accounts of the structure of propositions, for not all elements of propositions could be classified under the usual headings. To see this, consider the following statement:

A child left too long in a car with the windows closed on a hot day is at risk of dying from heat stroke.

Here we see elements such as [too], [in], [the], [with] [at], [of], [from], which are neither singular nor general terms, neither quantifiers nor the copula. Even if one subscribes to the view that all propositions can be expressed in subject–predicate form, it is clear from this example that the subject term and the predicate term may themselves be complex. The subject-term of the above proposition, for example, might be [A child left too long in a car with the windows closed on a hot day]. But no one had provided a theory determining just which complex terms (or concepts) could serve as subject or predicate. The received view on term meaning (set out in the highly popular Port Royal Logic and elsewhere) was that terms were all simple aggregates of characteristics, on the model of [male, caucasian, green-eyed, ...]. The above example shows this to be clearly inadequate, as does pretty much any mathematical

¹ Bolzano discusses attempted definitions in the **WL**, §23.

proposition. How, for instance, could the traditional view account for propositions such as those expressed by the following?

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\int_{-\pi}^{\pi} \sin x \, dx = 0$$

$$\frac{4+9}{5+3} < \frac{7+8}{1+4}$$

What are their subjects, what are their predicates, what rules licence us to form these combinations but not others? No good answers were forthcoming from the logicians.

Contemporary treatments of logic deal with such questions as a rule only for limited systems. They accomplish the task by specifying: (1) the basic symbols of an artificial language and (2) rules that determine which combinations of these symbols are to count as meaningful. Here is a simple example, describing a language capable of expressing some propositions of basic arithmetic. The symbols are as follows: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, +, =, (,). We then set out the following rules of combination:

1. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, are all *numerals*.
2. Any finite string of two or more occurrences of symbols drawn from the list 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, which does not begin with 0 is a numeral.
3. All numerals are *terms*.
4. If $\mathcal{A}$ and $\mathcal{B}$ are terms, then $(\mathcal{A} + \mathcal{B})$ is a term.
5. If $\mathcal{A}$ and $\mathcal{B}$ are terms, then $\mathcal{A} = \mathcal{B}$ is a *formula*.

We can see from these rules that ‘3’, ‘1239’, ‘54675’ are numerals, that ‘(7+5)’, ‘(3+(5+4))’, ‘((7+6)+((5+5)+4))’ are terms, and ‘(7+6)=(8+1)’, ‘1=3241’, ‘3=(1+2)’, ‘(2+2)=(1+3)’ are formulae, while the strings of symbols ‘0001’, ‘2++3+’, and ‘=+35’ are none of these things.

Traditional logic was far more ambitious in its aims. The analysis of propositions it presented was supposed to cover not only limited systems, but the entire range of meaning. To do this properly, following the above model, it would have been necessary to produce an exhaustive list of basic elements (or

at least a complete enumeration of the different *kinds* of basic elements), as well as a complete set of rules of combination. If successful, the result would be something like a universal grammar, containing rules of combination not only for limited systems but for any possible system. The contrast between this lofty ambition and the toy theories set out in most logical treatises could not have been more striking.

Bolzano maintained the global approach of traditional logic, but abandoned the pretence of being able to give a complete map of possible forms of thought. It was in part for this reason that he began with the proposition rather than with ideas or terms. We saw above that he gave a preliminary characterization of propositions as abstract entities which are either true or false but not both. In saying this, he did not aim at giving a proper *definition* of the concept [proposition], that is, an indication of the parts of the concept and how they are combined, but rather a simple orientation or explication (*Verständigung*) permitting the reader to figure out what he uses the term to refer to. In addition to being true or false, propositions are complex entities, that is, they have *parts*. These parts, provided they are not themselves complete propositions, he calls *ideas*. He thus, at least provisionally, takes the concept [proposition] as primitive, or undefined, and defines the concept [idea] in terms of it.¹ Bolzano's decision here is a subtle change from received views, but as we shall see it had far-reaching consequences. Because of it, he was able to work out his logical theories without having first developed a definitive analysis of the structure of propositions.

Perhaps it was a mistake, as Bolzano's correspondent Exner suggested, to define ideas as parts of propositions, the part in terms of the whole? In a letter of 1834, Bolzano responds to this objection:

...[T]he concept of a complex object is by no means always composed of concepts of its parts. Thus, for example, the concept of a clock is composed of the concepts "machine", "time", "measure", but not of the concepts "dial", "hand", etc. So the *concept* of a proposition in the objective sense of the word, if it is not completely simple, might well be composed of other components than those you suppose when you think of the components of which the *proposition in itself* is composed. As concerns the concept of an *idea* in particular, I am inclined to believe that the explication "an idea is a part of a proposition that is not itself a

¹ **WL**, §48, no. 2 [I.216]. Later (§128 [II.18]) he indicates that he is strongly inclined to regard his definition of [idea] as correct.

complete proposition” is the true *definition* of this concept. For it is quite common for the concept of a part to include the concept of the whole. Thus for example, in the concepts of dial, hand, etc., the concept of the entire clock surely occurs. For we certainly think by dial simply “a surface inscribed with numbers in such a way that it can be used in a *clock* in order to ... etc.” So too the concepts “head”, “neck”, “heart”, “lung”, etc., cannot be defined without mentioning the relations in which these parts stand to the whole organism; i.e., in the concept of the part there occurs the concept of the whole that the part belongs to.¹

It had always been maintained that ideas (or terms) were, or at least could be, parts of propositions, but before Bolzano it had never occurred to anyone to use this attribute to define the concept [idea]. One important consequence of this decision was that it drew attention to a number of elements of meaning (also ideas according to Bolzano’s definition) that had been widely overlooked by modern, if not by medieval, logicians, namely, those designated by words such as ‘and’, ‘not’, ‘the’, ‘which’, and the like, the so-called syncategoremata.

5. THE NATURE OF IDEAS

Looking upon ideas as parts of propositions also focused attention on the role ideas play within propositions. In the traditional way of looking at things, because ideas came first, most attention had been paid to the relation between an idea and its object(s), and numerous philosophers had advanced bold speculations on the nature of this relation. Often, it was conceived in terms of resemblance, the idea (or mental copy) being seen as the image of its object (the original), much like a photograph archived in the mind. Here Locke:

[M]ethinks the *Understanding* is not much unlike a Closet wholly shut from light, with only some little openings left, to let in external visible Resemblances, or *Ideas* of things without: would the Pictures coming into a dark Room but stay there, and lie so orderly as to be found upon occasion, it would very much resemble the Understanding of a Man, in reference to all Objects of sight, and the *Ideas* of them.²

¹ Bolzano to Exner, 22 November 1834 [BBGA 3.4/1, p. 87; MM-EX, p. 145].

² *Essay*, II, xi, §17.

How such pictures were supposed to function in judgments, or how they were supposed to fit together to create something that could be true or false, remained a mystery, concerning which little if anything of substance was said.

Consider, for example, a simple judgment like [All whales are mammals.]¹ One way to understand this statement is to see it as a claim about *classes*, *sets*, or *extensions*. It might then simply be taken to say that the class of whales is contained in the class of mammals, or that the extension of the idea [whale] is a subset of the extension of the idea [mammal]. On this interpretation, the idea [whale] is here called upon to represent the class of whales. If the meaning of this term is an idea in Locke's sense of the term, thus an image, then an image must somehow do this work. But how? Mere resemblance cannot explain this, for resemblance is symmetric: if the copy resembles the original, the original by the same token resembles the copy, as well as any other copies that may happen to exist (e.g., other people's mental images of whales). Nor could it be simply a matter of cause and effect, since I can think of whales under the most varied of circumstances, most, perhaps even all of them not graced by the presence of such a creature. The idea [whale], moreover, is general, while images seem not to be, a discrepancy which gave rise to not a little perplexity among the adherents of picture theories of ideas.²

Bolzano does not seek to explain in general how it is that ideas have objects.³ Indeed, as he remarked in a notebook, he thinks it cannot be explained: "...The question: what makes a certain idea *X* an idea of a certain object [...] cannot be answered."⁴ He made a similar point in a paper read to the Bohemian Royal Society in 1843:

[Th]e concept in question is among those that can scarcely be analyzed into parts: it is either an entirely simple concept, or else

¹ We will use square brackets to form designations of subjective ideas and propositions as well as designations of the corresponding objective entities, relying upon context to disambiguate.

² See, for example, Locke's discussion of abstract ideas (*Essay*, III,iii; IV, vii, §9, and elsewhere) and Berkeley's criticisms (*Principles*, Introduction).

³ In the case of the *subjective* ideas called *intuitions*, Bolzano does offer an explanation of the relation, namely, that the object of the idea is its cause. See below, pp. 227 ff. Obviously, though, this has no application in the case of ideas *in themselves* of any kind, since these, lacking actuality, have no causal relations.

⁴ "Verbesserungen und Zusätze zur Logik" [BBGA 2A.12/2, p. 135]: "...die Frage, was macht eine gewisse Vorstellung *X* zu einer Vorstellung von einem gewissen Gegenstande [...] läßt sich nicht beantworten." Cf. L. Wittgenstein (*Phil. Untersuch.*, 3rd edn [Oxford: Blackwell, 2001], p. 151): "Was macht meine Vorstellung von ihm zu einer Vorstellung von *ihm*? Nicht die Ähnlichkeit des Bildes."

consists of a couple of parts for which we possess no linguistic designation. “To be the object of an idea” or, what amounts to the same thing, “to be represented” is something so *particular* and at the same time so *simple*, that right at the start, I confess, I lost all hope of using a combination of two or more concepts to indicate what it is.¹

To represent an object it is neither necessary nor sufficient for an idea to resemble it. Many logicians, by considering only certain ideas, had found the resemblance theory plausible enough despite its problems. Had they simply paid attention to the large variety of ideas (i.e., parts of propositions) there are, Bolzano remarked, they would have been immediately convinced of its wild implausibility:

[W]hoever compares *being represented* with a kind of *depiction*; whoever wants to speak as if the object were related to its idea in roughly the same way that a *sensible* object is related to its *picture*; whoever merely assumes that there is a certain *similarity* between *idea* and *object*, an *agreement* between their respective attributes; especially, however, whoever (like our Philosophers of Identity) speaks of the complete *equality*, indeed *identity* of the two; such a person, I say, has already embarked upon the most dangerous path, one which, should he continue to wander along it, will keep his back forever turned to clear and distinct thought. For tell me what kind of *similarity* obtains, or could obtain, between the idea *something* on the one hand, and its objects on the other, i.e., each and every thing there is? Show me a pair of things *more dissimilar* than the ideas *attribute*, *spatial object*, *tool*, *concept*, *proposition*, *etc.* and the objects that are, respectively, represented by these ideas.²

The resemblance theory is built upon the assumption that ideas are “that which [the] mind is applied about whilst thinking,”³ another view that Bolzano found profoundly mistaken. While he agrees that ideas are sometimes

¹ “Aufsatz, worin eine von Hrn. Exner in seiner Abhandlung: ‘Über den Nominalismus und Realismus’ angeregte logische Frage beantwortet wird” (Prague, 1843) [BBGA, 1.18, pp. 71–8, p. 74. MM-EX, p. 183].

² BBGA, 1.18, p. 74. MM-EX, p. 183.

³ Locke, *Essay*, II, i, §1. Cf. I, i, §8, where Locke says that the term “Idea ... serves best to stand for whatsoever is the Object of the Understanding when a Man thinks.”

the object of our thought (namely, when we are thinking about ideas), he steadfastly denies that this is what occurs when we think about objects other than ideas. To think of, say, an individual dog is not to think about or perceive an idea of that dog. Rather, it is to have an idea that has that dog as its object. Even when I do have an idea before my mind, as the object of my thought, Bolzano maintains, this does not occur in the way Locke suggests. Rather, in order to have one idea as the object of my thought, I must have another idea, namely, an idea of the former idea. In any case, ideas are not images. To think so is a disastrous mistake. An idea is not a picture,

...not an object that we examine in place of another. Rather, it is what arises in our mind when we examine the object itself.¹

One might still maintain that ideas had to resemble their objects, but this assumption would be entirely gratuitous, like a man who keeps his corkscrew in a drawer insisting that you need a desk to open a bottle of wine.²

In presenting his own account of the relation between ideas and their objects, Bolzano takes the time to refute two commonly held views. According to the first, the parts of an idea are ideas of the parts of its object. In his *Logic*, for instance, Kant had claimed that an intuition (a kind of idea) of a house must contain representations of doors and windows, etc.³ Bolzano quotes Abicht, a Kantian logician, as saying: “The concept of an object must allow us to distinguish as many parts of ideas in it as can be differentiated in the object of this concept.”⁴ It seems clear that this view was held in the

¹ **WL**, §52, no. 5 [I.231].

² Benno Kerry, who knew Bolzano’s writings well, appreciated this point nicely: “...the relation between concept and the object of a concept may well belong among the original, irreducible relations. This relation can go hand in hand with that of similarity, and indeed, as we have seen, with that of the equality of a concept and its object. But it cannot be said that this relation is exhausted by those of similarity or equality. This follows already from the fact that one concept may be just as similar to another, differing perhaps only by being less determinate by one characteristic, as it is to its object—yet no one would confuse the relation between two such concepts with the relation between a concept and its object. That in the relation between concept and the object of a concept which cannot be reduced to similarity or equality is precisely what is characteristic of this relation, namely, the moment of a peculiar belonging of the object to the concept” (“Ueber Anschauung und ihrer Psychische Verarbeitung,” *Vierteljahrsschrift für wissenschaftliche Philosophie* **10** [1887], p. 460).

³ *Logik*, ed. Jäsche, Introduction, V.

⁴ Abicht, *Verbesserte Logik* (Fürth, 1802), p. 362; **WL**, §63 [I.267].

main by those philosophers who thought of ideas as images of their objects, mental photographs, if you will. For Bolzano, who defined ideas as parts of propositions, this view was untenable. Consider, for example, ideas such as [a country without mountains] or [a book without illustrations]. These ideas contain the ideas [mountains] and [illustrations] as parts, yet clearly these are not ideas of parts of their objects. The same holds for ideas such as [the eye of the needle] or indeed, on Bolzano's understanding, the idea of an idea, i.e., [part of a proposition which is not itself a proposition]. In these ideas, we find as parts ideas of a needle and a proposition, which once again are not ideas of parts of their objects. This is even more clear in the case of *objectless* ideas. The parts of the ideas [round square] or [greatest prime number], for example, cannot be ideas of parts of the objects of these ideas, for there are no such things.

According to the second thesis, "the idea of an object must contain ideas of all the attributes of the object as its parts."¹ This view was held, notably, by Arnaud and Nicole, the authors of the highly influential Port Royal Logic. These authors had defined the *comprehension* of an idea as follows:

The comprehension of an idea is the constituent parts which make up the idea, none of which can be removed without destroying the idea. For example, the idea of a triangle is made up of the idea of having three sides, the idea of having three angles, and the idea of having angles whose sum is equal to two right angles, and so on.²

In case there is any doubt about whether they held that every idea contains ideas of all the necessary attributes of its object(s), the authors provide the following example:

In a sentence expressing an affirmation the entire comprehension of the idea expressed by the predicate must be contained in the comprehension of the idea expressed by the subject. For example, when we say 'A rectangle is a parallelogram,' we mean that the whole idea of parallelogram is contained in the idea of rectangle. Were there any part of the idea of parallelogram which failed to be part of the idea of rectangle, then we could not affirm the idea of parallelogram of the idea of rectangle but would have

¹ WL, §64 [I.269].

² Arnaud and Nicole, *Logic, or the Art of Thinking* (1662), tr. J. Dickoff and P. James (Indianapolis: Bobbs-Merrill, 1964), Part I, Chapter 6.

to deny the one idea of the other. This principle of affirmation is the basis of all affirmative arguments.¹

Bolzano rejects this highly implausible view as well. Were this the case, he remarks, many if not all ideas would have to contain infinitely many parts. The idea $[\sqrt{2}]$, for instance, would have to contain as parts each coefficient in the decimal expansion of this number (since these certainly reflect attributes of its object). Bolzano finds it doubtful that humans could grasp such ideas. A still more convincing counterexample, he thinks, may be found in the idea [equilateral triangle]. The objects standing under this idea all have the attribute of being equiangular as well as equilateral. But there is no reason to suppose that the idea [equiangular] is part of the idea [equilateral triangle]. For this latter idea is formed by appropriately connecting the ideas [triangle] and [equilateral]. And, if the proposed theory were true, the concept [equiangular] could not occur as part of either one of these ideas. For if it did, then it would have to occur as part of one or the other. But it cannot be part of the idea [triangle], since not all objects standing under that idea are equiangular. On the other hand, if [equiangular] were a part of the idea [equilateral], then the concept [equilateral *quadrangle*] would contain it, and thus apply only to equilateral, equiangular quadrilaterals. But again, not all equilateral quadrangles have equal angles.²

(A) IDEAS AND THEIR OBJECTS

Bolzano gives two different definitions of the extension of an idea. In the essay on mathematical method, he writes that the extension of an idea is simply the collection of objects the idea represents.³ In the *Theory of Science*, however, he writes:

By indicating the objects to which a certain idea applies, we indicate the *range*, *extension*, or *sphere* of this idea. By these expressions I understand that particular attribute of an idea by virtue of which it represents only those and no other objects.⁴

The formulation of the *Theory of Science* seems the more careful of the two, and it probably better reflects Bolzano's considered opinion—the additional

¹ Arnould and Nicole, *Logic*, Part II, Chapter 17.

² **WL**, §64.

³ **ML**, §5 [BBGA 2A.7, p. 53; **MM-EX**, p. 46].

⁴ **WL**, §66 [I.297–8].

complexity it involves, we may conjecture, he thought well to omit in a summary of his logic intended for mathematicians.

Why does Bolzano consider the extension of an idea to be one of its attributes, rather than simply the set of objects it represents? The most important reason, we think, is that extensions serve in many cases to individuate ideas, so there must be some feature *of the idea* which distinguishes it from others in this respect. It would seem odd to say that the objects themselves were a part or attribute of the idea, however—for the objects may be actual, while the idea in itself never is, and hence it would seem that none of its parts or attributes can be either.

One peculiarity of Bolzano's account should be noted here. Because he did not accept the notion of an empty collection, he could not say that ideas without objects, such as [round square], have (empty) extensions. Instead, he says that they have no extension at all.

In claiming that the extension of an idea was one of its attributes, Bolzano was again in disagreement with a number of modern philosophers, those who maintained that ideas in and of themselves bore no relation to objects. In the *Treatise*, for example, Hume had written:

To form an idea of an object, and to form an idea simply is the same thing; the reference of the idea to an object being an extraneous denomination, *of which in itself it bears no mark or character*.¹

In a letter to Bolzano, Franz Exner reflected a similar view of the matter:

Let *a* be a simple singular idea in a subject. It is a determinate state of consciousness, aroused by a certain object. That this idea is applied by the subject to a certain object is completely accidental to the idea itself.

In my opinion children originally apply such states of consciousness to nothing at all; adults often apply them to the wrong objects and must afterwards correct themselves. However, they do this in any case (whether correctly or incorrectly applied) only by means of another idea that is present at the same time, which enters into combination with *a*. Supposing that in a certain soul merely the idea *a* were present, it would apply to no object at all. For this subject it would be an objectless idea²

¹ *A Treatise on Human Nature*, I.i.7, emphasis added.

² Exner to Bolzano, 11 November 1834 [BBGA 3.4/1, p. 79; MM-EX, pp. 137–8].

For Exner, a subjective idea only gains an object once a thinking subject *applies* it to that object. Bolzano argued that this concession amounted to a *reductio ad absurdum* of such theories of ideas. For the best sense he could make of the locution ‘the idea *A* is applied to an object’ was to interpret it to mean that [*A*] occurred as the predicate in a judgment, the subject-idea of which, [*X*], represented the object in question, i.e., a judgment of the form:

[*X* is *A*.]

But to form such a judgment, we must already have an idea [*X*] which represents the object—and it is precisely the possibility of such an idea that we are supposed to be explaining.¹

(B) THE INDIVIDUATION OF IDEAS:
EXTENSION, CONTENT, MODE OF COMBINATION

By looking upon ideas as parts of propositions, and especially by looking upon ideas in themselves as parts of propositions in themselves, Bolzano avoided these and related problems. By confining his attention to abstract objects, ideas in themselves, he removed the temptation to think of ideas as a natural kind that could simply be pointed out, as one might point out a fish or a fern, pending further investigation of their attributes. Instead, he had to specify exactly which attributes these abstract objects had, always seeking, as good method prescribes, to ascribe to them the bare minimum of properties and relations required to support the claims made about them.

In the case of ideas in themselves, he attacks this problem by looking at their logical function. What contribution, he asks, does this part of the proposition make to the whole? In the proposition [All whales are mammals], for instance, the term [whale] seems to have the function of referring to the class of whales, or of furnishing an extension. This is what the idea does, and this is also pretty much what the idea *is*. Thus the *extension* of this idea is, if not the only, at least one of its essential attributes.

The hesitation in the last sentence is due to the observation that different ideas may have the same extension. Thus the clearly different ideas [the man who was president of the USA after Nixon] and [the man who was president of the USA before Carter] represent the same man (Bolzano calls ideas that have the same extension *interchangeable* or *equivalent*,² while today we

¹ Cf. WL, §66.

² WL, §96.

would call them *coextensive*). Partly in order to account for such differences, Bolzano introduces the concept of the *content* of an idea. This he defines as “the sum of the parts of which a given idea consists.”¹ The technical term ‘sum’, which Bolzano uses here, indicates among other things that the way the parts are combined (their *Verbindungsart*) is not determined by the content.² This observation is important, since under his definition ideas may have the same content (since they have the same ultimate parts) and yet differ, as the following examples show:³

[learned son of an ignorant father], [ignorant son of a learned father]
[3⁵], [5³]

That the way the parts of an idea are combined is alone sufficient to distinguish ideas in certain cases is brought out with the help of a clever example⁴ of a pair of ideas that have the same content and extension but are nevertheless distinct, namely, [2⁴] and [4²].

What Bolzano says here seems to be correct as far as it goes, but there are a couple of important oversights, which become particularly apparent in the case of what we have been calling syncategorematic ideas, ideas such as [and], [or], [not], (recall that for Bolzano an idea is any part of a proposition which is not itself a proposition). Bolzano claims that the ideas [has] and [not] are both simple and objectless.⁵ Thus these ideas do not differ in extension (they have none), nor in content (again, they have none—for where there are no parts, there is no sum of parts either), nor in the arrangement of their parts (again because there are no parts). They differ, but Bolzano has no account of how they do.

One way we might distinguish the ideas [not] and [has] is by noting that they cannot fit into the same places in propositions—in a sentence, replacing ‘has’ by ‘not’, or vice versa, generally if not always results in nonsense.

¹ WL, §56.

² WL, §96, no. 2; cf. *Paradoxien des Unendlichen*, §4. Following the precedent of translating Frege’s expression ‘*Art des Gegebenseins*’, as ‘mode of presentation’, one might also render ‘*Verbindungsart*’ as ‘mode of combination’. Bolzano’s notion of a sum is discussed below, Chapter 8, pp. 430 ff.

³ WL, §56.

⁴ WL, §96, no. 2.

⁵ The claim of simplicity is made at WL, §89, no. 1 [I.415] and ML, §3, no. 2 [BBGA 2A.7, p. 49; MM-EX, p. 42]; of objectlessness at WL, §78, note 2 [I.360]. Cf. §118, no. 1 [I.558].

Consider, for example:

Socrates has wisdom. / * Socrates not wisdom.

2, which is prime, is not even. / * 2, which is prime, is has even.

Husserl would later give the name of *semantic categories* to this feature of meanings, whereby they are supposed to divide into classes such that any member of a given class may be substituted for any other without loss of meaningfulness.¹ We shall see later (pp. 294 ff.) that Bolzano had some inkling of this notion, even if it wasn't distinctly discussed in his chapter on ideas.

But even semantic category is not enough to do what is required here. For suppose, as seems at least possible, that we have two syncategorematic ideas such as [or] and [and] which belong to the same semantic category and which are both simple. In this case, we would still have no way of distinguishing them on his account.²

(C) THE LOGIC OF CLASSES

Bolzano defines a number of relations between the extensions of ideas, thus providing the concepts for an elementary logic of classes.³ Ideas $A, B, C, D, \dots$ are said to be *compatible* iff they have an object in common (otherwise *incompatible*). Thus, for example, the ideas [even number] and [prime number] are compatible since 2 is represented by both of them, while [cow] and [prime number] are incompatible. An idea B is said to *include* an idea A iff all the objects of A are also objects of B . If the inclusion is strict (B contains A but A does not contain B), A is said to be *subordinate* to B (example: [naked mole rat] is subordinate to [mammal]). When A includes B and B includes A , the two ideas are said to be *equivalent* or *interchangeable* (example: [third president of the United States of America] and [author of the Declaration of Independence]). If A and B are compatible, but neither includes the other, they are said to be *overlapping* (example: [regular polygon] and [triangle]).

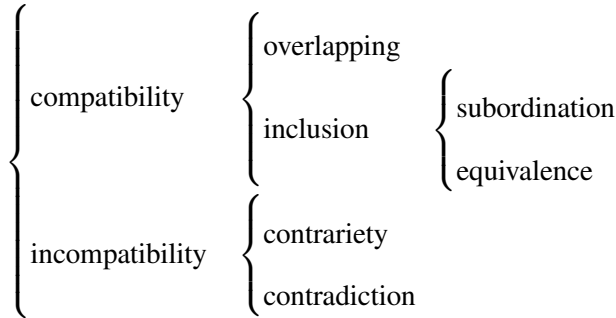
¹ *Logical Investigations*, IV, §10. Cf. S. Centrone, *Logic and Philosophy of Mathematics in the Early Husserl* (Dordrecht: Springer Verlag, 2010), section 2.6.

² In his essay "Function and concept," Frege shows one way to distinguish them—namely, by considering them to represent different *functions*, which can, in a certain extended sense, also be assigned extensions. Incidentally, Bolzano did not think that the concept expressed by 'or', at least when it occurs as a sentence connective, was simple. See below, p. 267.

³ **WL**, §94 f.; **ML**, §5 [**BGA** 2A.7, pp. 53–7; **MM-EX**, pp. 46–9].

If an object is *A* if and only if it is not *B*, *A* and *B* are said to be *contradictory* (otherwise, incompatible ideas are called *contrary*) (examples: the ideas [something which is red] and [something which is not red] are contradictory, while [something which is red] and [something which is green] are contraries). We can depict Bolzano's classification of these relations with the following diagram:

Relations between the extensions of ideas



Bolzano proves a fair number of theorems concerning these relations (WL, §105), but there is little noteworthy or original in this part of his treatment.¹ In §108 of the *Theory of Science* and §5 of “On the Mathematical Method”, however, he does take a new step when he seeks to extend these relations to objectless ideas. Consider the ideas [regular solid with more than 20 faces] and [regular solid with 21 faces]. Bolzano remarks that many logicians would find it acceptable to say that the latter idea is subordinate to the former even though neither has any objects. In order to make sense of this usage, he suggests that we apply the following method, reminiscent of Bas van Fraassen's method of supervaluations, to extend the concept of subordination to such ideas:

[W]e consider certain of their components *i, j, ... variable*, and pay attention to the behaviour of the infinitely many *new* ideas that are produced when we replace *i, j, ...* with different ideas, noting what occurs whenever one or the other of them becomes an objectual idea. We say that *A* is *higher* than *B* if on every occasion when a certain determination of the variable parts

¹ J. G. E. Maaß (*Grundriß der Logik* [Leipzig 1793; there were numerous later editions]), as Bolzano remarks, had defined a similar set of relations. Cf. J. Sebestik, *Logique et mathématique chez Bolzano*, pp. 171 f.

$i, j, \dots$ for which A or B becomes an objectual idea, A represents all the objects of B as well as some others.¹

In the case under consideration, we can vary the part [regular]. In every case where a substitution (e.g., [three-dimensional]) produces new ideas that have objects, we find that the latter is subordinate to the former. This technique is especially useful, Bolzano suggests, in dealing with objectless concepts in mathematics such as $[\sqrt{-1}]$. (For Bolzano, the expression ' $\sqrt{-1}$ ' designated an idea of a *quantity* that, multiplied by itself, yielded -1 , and there was no such quantity. He also, by the way, held the idea [zero] to be objectless on similar grounds). Thus, for example, he thinks we should consider equations (which he construes as expressions of the equivalence of ideas) such as

$$\sqrt{-2} \cdot \sqrt{-3} = \sqrt{-2 \cdot -3}$$

as being verified in this extended sense (i.e., since $[\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}]$ is true whenever $[a]$ and $[b]$ are chosen so as to produce objectual ideas). It should be noted that the extended logical relations spoken of in such cases are relative to a choice of components that are to be varied in the ideas in question, a constant feature of his use of this technique of variation.

(D) SIMPLE AND COMPLEX IDEAS

A basic thesis of Bolzano's ontology is that every complex whole is ultimately composed of simple parts, parts that themselves have no parts. So too in the realm of logical objects, in particular with respect to ideas. Though there are undoubtedly complex ideas, each is, he maintains, composed ultimately of simple ideas.² Having no parts, simple ideas have no content strictly speaking, though using the word in an improper sense, one could say

¹ **ML**, §5, no. 7 [**BBGA** 2A.7, pp. 56–7; **MM-EX**, p. 49].

² **WL**, §61. Bolzano mentions geometrical objects as an apparent counterexample to the claim that all objects are composed of simple parts. Lines, for example, are said to be infinitely divisible, and at the same time it is claimed that every part of a line is itself a line, which therefore has further parts. Bolzano's response is that if the word 'part' is understood in Euclid's sense, i.e., to mean the same as 'homogeneous part', the remark is certainly correct, but irrelevant. For on his view all geometrical objects are ultimately composed of simple, though heterogeneous, parts, namely, points. In the case of ideas in themselves, however, Bolzano also supposes that the simple parts are homogeneous with the wholes—they too are assumed to be ideas. Cf. "Über den Begriff des Schönen," **BBGA**, 1.18, p. 101: "For what else should a [complex] idea consist of if not other ideas?"

that the content of a simple idea is just the idea itself.¹ Clearly, too, where there are no parts, there can be no particular way of arranging the parts. As extension appears to be the only remaining feature that would allow us to distinguish ideas, it seems to follow that there cannot be two distinct simple ideas (ideas in themselves, that is), with the same extension:

If [...] one were to ask whether two *simple* ideas could be interchangeable, the answer would have to be no. For in order to distinguish two things one must be able to assert different things about them. But it seems at least that all assertions about an idea concern one of two things: either the objects it represents, or the idea itself; and with respect to the latter, whether it is simple or complex, and if complex, what parts it is composed of, etc. Now interchangeable ideas cannot be distinguished with respect to their objects, since they apply to the same ones. Thus it seems that one can only distinguish them either by declaring one of them to be simple and the other complex, or declaring them to be composed of different parts, or finally, declaring them to be composed of the same parts, but combined in different ways. But if both ideas are simple, we would be able to indicate no difference between them, and could hardly look upon them as being different.²

Simple ideas with objects, then, are individuated solely by their extensions.

(E) OBJECTLESS IDEAS

Not all ideas have objects, however. Bolzano recognizes several kinds of these.³ First of all, there are ideas such as [round square], which would at-

¹ WL, §92.

² WL, §96, no. 3 [I.446].

³ Bolzano's defence of the view that there are objectless ideas gave rise to intense speculation among the students of Brentano, notably Twardowski, Husserl, and Meinong, for it seemed to contradict a central claim of their teacher, who had maintained that all ideas or presentations, indeed all mental phenomena, are intentional, directed towards objects. See, e.g., Kasimierz Twardowski, *Zur Lehre vom Inhalt und Gegenstand der Vorstellungen* (Vienna, 1894), §5 (Eng. tr. by R. Grossman, *On the Content and Object of Presentations* [The Hague, 1977]). Also Edmund Husserl, "Intentionale Gegenstände" (unpublished MS, 1894), pp. 303–48 in *Husserliana*, Vol. XXII (The Hague, 1979); Alexius Meinong, "Über Gegenstandstheorie," pp. 1–50 in A. Meinong, ed., *Untersuchungen zur Gegenstandstheorie und*

tribute incompatible properties to an object. He calls such ideas *imaginary*.¹ Second, there are complex ideas which contain no internal contradiction, but rather lack objects for some other reason, for example, [female twentieth-century US President]. A third kind includes ideas that don't even seem to be of a kind which could ever represent objects, ideas such as [has], [not], [and], corresponding to syncategorematic terms. Let us deal first with the first two kinds of ideas. In many cases, as the first two examples show, objectless ideas are complex, but have parts (e.g., [square], [president]) that do have objects. Perhaps, as Aristotle had suggested, only complex ideas could be objectless? Bolzano seems to disagree, as he claims that the third kind of ideas mentioned above does include some that are both simple and objectless, for example, the ideas [and] and [has].² But these ideas, as mentioned above, do not seem to be the sort that can have or lack objects, so one might still wonder whether, on Bolzano's view, among the sort of ideas that can properly be said to have or lack objects (*denoting* ideas, to adapt Russell's terminology), any are both simple and objectless. To our knowledge, Bolzano never raised this question, though it does seem to follow from what was said above that there could be at most one simple, denoting, objectless idea.³

Bolzano's distinction between objectual and non-objectual ideas caught the attention of Alain de Libera, a French specialist of medieval philosophy. In *L'art des généralités*, he finds affinities between Bolzano and Avicenna. We leave for another occasion the analysis of this fascinating discussion and quote just the concluding lines.

[Avicenna's] distinction between two kinds of *ma'nā-s*, "existing in the soul"—those which "refer to something exterior" and those which "do not refer to something exterior"—is a rather good equivalent of the distinction between "objectual" and non "objectual" concepts and, evidently, its function is to protect ontology from an inappropriate reduction of "objectless representations" to "representations of objects which do not exist". [...] Avicenna's thesis is that a nature can be authentically aimed at, or, if you prefer, that there is an objectual concept of a nature

Psychologie (Leipzig, 1904); Eng. tr. pp. 76–117 in R. Chisholm, ed., *Realism and the Background of Phenomenology* (New York, 1967).

¹ WL, §70.

² See above, p. 218.

³ He does say that he believes the idea [nothing] to be complex, analyzable as [not something] (WL, §89, no. 3 [I.417]). Though not decisive, this suggests that he might have thought that there are no simple, denoting, objectless ideas.

only if, as regards existence, this nature is abstract in sense /Ab1/ [i.e., if it is taken without the condition that some other thing is with it]. Abstraction with respect to existence (in the sense of /Ab1/) is the most important, though the least known, part of Avicenna's legacy to scholasticism. [...] The second part, in agreement with the preceding, consists in showing the referential character of concepts. An "objectual" concept is a concept that refers to something [...] This is why Avicenna's true interlocutors are neither Lychetus nor François Antoine of Brindisi. Even if it involves leaping over centuries, we prefer others—such as Bolzano.¹

(F) INTUITIONS AND CONCEPTS

This is a good place to raise another important question: can a simple idea be at the same time a *singular* idea, that is, an idea with just one object? The weight of logical tradition said no: simple ideas were all claimed to be *general* ideas, ideas which had (or at least could have) more than one object. What basis was there for this claim? One popular view was that simple ideas were all (ideas of) *characteristics* or properties of objects, such as [red] or [cold].² Support for this view often came from the assumption that ideas were mental states that could be repeatedly provoked by the action of external objects. I see red today, I can see red tomorrow, and the day after; so too with pain, sounds, or the smell of sauerkraut. If now I identify the experience with the idea (to have the idea [red] is just to see red), and in addition I am somewhat ham-handed with the concept of identity (so that the experience of seeing red I had yesterday and the one I had today are said to be the *same* idea), then, obviously (!), all ideas are general. So, at least, went one line of thought on the question.

The assumption that all simple ideas are ideas of characteristics was also the basis of the claim that the content and the extension of ideas vary inversely: the more an idea contains, the less it represents. This happens because adding more characteristics to an idea strengthens the condition it expresses, thus making it harder to meet. For example, the idea [white, round,

¹ *L'art des généralités* (Paris: Aubier, 1999), pp. 605–7.

² In a semantic confusion common at that time, the word 'characteristic' (*Merkmal* in German) was used to refer both to properties of objects *and* to the ideas that represent these properties.

cold] applies to fewer objects than the idea [round, cold], which in turn applies to fewer objects than [cold]. To represent a single object, therefore, highly complex ideas would be required, perhaps even, as Leibniz had suggested, infinitely complex ideas. A simple idea, it seemed, could never do this.

It is clear that arguments such as those just sketched have no force whatsoever concerning Bolzano's ideas *in themselves*, which are neither mental events, repeatable, nor provoked by sauerkraut. With his richer conception of the structure of ideas, Bolzano also saw the contention that all parts of ideas are (ideas of) characteristics for what it was: an arbitrary supposition maintained only by force of habit. Insofar as the claim that all simple ideas must be general ideas depended upon the view that ideas are mere sums of characteristics, it had no real support. The claim that the content and extension of ideas vary inversely is decisively refuted with a counterexample: [a man who speaks French or German], for instance, has a greater content *and* a greater extension than the idea [a man who speaks French].¹ Thus the claim that singular reference could only be accomplished through (possibly infinite) complexity was seen in its turn to lack support.

It seems clear that no good reason had been given to deny that there are simple ideas *in themselves* with exactly one object. Borrowing terminology from Kant and his followers, Bolzano suggests that we call such ideas *intuitions*.² *Concepts*, in turn, are defined as ideas in themselves that are not intuitions and contain no intuitions as parts, while *mixed ideas* contain both concepts and intuitions.

The above arguments might succeed in opening a space for simple, singular ideas as an abstract possibility, but Bolzano was not content to leave the matter there. Instead, he argued quite persuasively that in the realm of *subjective* ideas there are those that are both simple and singular and that, moreover, the existence of such ideas was an unrecognized presupposition of most modern theories of perception. And since he maintained that for every subjective idea there is a corresponding objective one,³ proving the existence of subjective intuitions would also establish the existence of objective ones.

¹ Letter to Exner, 18 December 1834 [BBGA 3.4/1, p. 102; MM-EX, p. 159]; Cf. WL, §120.

² Kant had reserved the term 'intuition' for singular ideas which stand "in immediate relation to their objects" (*Critique of Pure Reason*, A 320). For a comparison of the theories of Kant and Bolzano, see R. George, "Intuitions, the theories of Kant and Bolzano," in *Semantik und Ontologie*, ed. M. Siebel and M. Textor (Frankfurt: Ontos Verlag, 2004), pp. 319–54.

³ WL, §271.

Perception, following the accounts most popular at the time, was supposed to work more or less as follows: objects outside the mind somehow affect it, causing a variety of mental states (called sensations, or sometimes ideas). Thus we might see various shapes and colours, smell a certain odour, hear various sounds, perhaps end by feeling pain when a moose is nearby. The mere presence of these sensations in the mind is not enough to assure us of the existence of an external object, however, as the phenomena of imagination, dreams, and hallucinations indicate. This, it was alleged, shows us that the moose is not directly present to consciousness. Rather, the mind must *judge* the occurrence of certain sensations to be caused by something external to itself.

Not questioning the general approach of such theories, Bolzano undertook a detailed investigation of the judgments they claimed to be involved in perception. At least some of these judgments assert that a certain sensation was caused by some extra-mental object. The *topic* of such a judgment is a single sensation, for instance, a pain I happen to feel at a certain moment, and what is claimed of it is that it was caused by something outside the mind. On a straightforward subject–predicate analysis—and this was the standard approach—the subject-idea of this judgment must represent the sensation. Note that we say *represent* here: we should not, Bolzano claims, think that the sensation is itself an idea. For it is by no means clear how a sensation like a pain could be part of a judgment, nor how it could represent anything, even if we were confining ourselves to subjective ideas. To say that a pain could be an objective idea would be still more implausible. A pain may be the *topic* of a judgment, but it is not an *element* of the judgment; it is the *object* of an idea, but is not itself an *idea*: “The sensation itself is one thing, the idea of the sensation quite another . . .”¹

It would be difficult to exaggerate the importance of this observation, or the difficulty that Bolzano’s contemporaries had in grasping it and its implications. For habitually this “semantic” layer was entirely absent from philosophical theories of meaning—ideas, the meanings of words, were thought, for example, to be at the same time sensations, or the fainter traces left behind by sensations, *and* parts of judgments. Yet there was no coherent account of how anything could fulfil both of these roles, how an assemblage of sensations and the like might have a truth value, and so on.²

¹ WL, §35, no. 8 [I.163].

² Cf. R. George, Editor’s Introduction to *Theory of Science* (Blackwell: London, 1972), pp. xxxix ff.

Now what is the nature of the subject-idea in such basic judgments of perception? It cannot, Bolzano observes, be a general idea like [pain], since the judgment does not treat of all pains, or of pain in general, but rather only of a single one, namely, *this* pain, the one I feel right now. In a letter to Exner, Bolzano explains:

I admit that *many* ideas that are produced in us on the occasion of the activity of an outer object on us have such *generality*, by means of which they represent not one but several objects. Of this kind are such ideas as “something”, “red”, “pleasant smelling”, “pain”, etc. But how can this be used to prove that *all* the ideas that appear in us on such occasions have this property? How could you show that alongside such ideas no others are produced that are not simply *applied* by us to a single object but rather only *represent* a single object? Were there no ideas of this last kind, true singular ideas, how could it come about that we nevertheless speak of a single outer object that acts upon us? How could we come to form the judgment: “*this* (this determinate feeling I have) *is a pain*”? ¹

Having established that basic judgments of perception involve singular subject ideas, Bolzano proceeds to argue that at least some of these ideas must be simple as well.

The first argument he offers is based upon a conjecture concerning the origin of such ideas. Bolzano claims that the idea designated by ‘this’ is the “next and immediate effect” of our attending to a change (i.e., the sensation) which takes place in the mind. The sensation, that is, is at the same time the object of the subjective idea and the cause responsible for its presence in the mind. Because it is the immediate effect of the change occurring in us, it must, Bolzano argues, be simple—and so too, consequently, must be the corresponding idea in itself. The intimacy of the connection between idea and object which occurs only in such cases is also thought to explain how it is possible for a simple act of the mind to refer to a single object.

¹ Letter to Exner of 22 November 1834 [BBGA 3.4/1, p. 90; MM-EX, pp. 147–8]; Cf. Bolzano’s letter to Exner, 9 July 1833 [BBGA 3.4/1, p. 27; MM-EX, p. 96]: “Someone brings a rose near us. We see—not red in general, but *this* red present in the rose. We smell, not odour in general, but rather only *this* pleasant fragrance, which just this rose has; when we are injured by one of its thorns, we feel not pain in general, but rather this determinate pain.”

As soon as we direct our attention upon the change that is caused in our mind by an external body, e.g., a rose that is brought before our senses, the *next* and *immediate* result of this attention is that the *idea* of this change arises in us. Now, this idea has an *object*, namely the change that takes place in our mind at that very moment, and nothing else. Thus, it has only one object and we can say that it is a *singular idea*. On this occasion other ideas, some of them no longer simple, are also produced by the continued activity of our mind; similarly, complete judgments are made, especially about the change itself that has just taken place. We say, for instance, “this (what I just see) is the sensation or idea ‘red’”; “this (what I just smell) is a pleasant fragrance”; “this (what I just feel upon touching a thorn with the tips of my fingers) is a painful sensation”, etc. It is true that in these judgments the ideas “red”, “pleasant fragrance”, “pain”, etc., have several objects. However, the ideas which occur in subject position and which we designate by the word *this* are certainly genuine singular ideas. For, by “this” we mean nothing but this individual change which takes place in us, and not a change that takes place elsewhere, no matter how similar it is to ours. Moreover, it is no less certain that all these ideas are also *simple*. For, if they were composed of parts, they would not be the *next* and *immediate* effect that results from the observation of the change that just now takes place in our mind; rather, several simple ideas, namely the parts of this complex idea, would have been generated earlier and more immediately.¹

This argument can be questioned on many counts: it relies on the unsupported speculation that the intuition is the immediate effect of the mental event, and it assumes that it makes sense to speak of an immediate effect of the mental event. It seems clear, as Rolf George has pointed out in an article on Bolzano’s intuitions, that the immediacy in question cannot be just a matter of temporal succession, since Bolzano maintains that in general many things happen in the mind at any given moment.² It is all the more unlikely that Bolzano meant “immediately following in time” here, given that he maintains that effects are simultaneous with their causes and knew full well that in a continuum (as he supposed time to be), it makes no sense to speak of the

¹ WL, §72.

² R. George, “Intuitions, the theories of Kant and Bolzano,” pp. 343–4.

next moment after a given one. But it is far from clear that he has a right to speak of another kind of immediacy in this connection.

Bolzano, however, provides a much better, and far more interesting, argument for the existence of subjective intuitions based on purely semantic considerations. Recall that the subject-idea of a basic judgment of perception is a singular term, having as its only object a single sensation, for instance, a pain I feel at some particular time. As a first approximation, we can try to express the subject-idea in the above judgment as ‘this pain’. Bolzano now asks: what role does the idea [pain] play here? In §59 of the *Theory of Science*, he proposes a surprising analysis:

A specially important kind of idea is that which we usually express in the form “this (or that) *A*”. I believe that this expression is taken in two different senses. In one, which I call the more exact sense, “this *A*” means roughly the same as “this, which is an *A*”. Here, the idea which is designated by the word “this” does not refer to any object other than the one that is represented by the complete idea “this *A*”, even if it is taken all by itself. The addition “which is an *A*” expresses an attribute which already belongs to the object associated with “this”, and is employed merely for the sake of greater clarity. In this sense we take an expression such as “This fragrance (which I just perceive) is sweet.” By the word “this” here, we mean the particular perception we have at this moment. That it is a fragrance is an attribute which already belongs to the object represented by “this”. Thus it is not the constituent “*A*” or “which is an *A*” which restricts our idea to the particular object that it has.¹

He claims, in other words, that the idea doing all the work here in singling out the object (the particular sensation of pain) is the one which we try, however ineptly, to express by means of the simple demonstrative ‘this’. The concept [pain], because it has more than one object, cannot represent just this one sensation, so the idea designated by ‘this’ must already do so (unless one adopts the implausible assumption that the idea [this] represents several objects, only one of which is a pain). But if this is the case, the concept [pain] is dispensable, (or, using Bolzano’s terminology, *redundant*): everything needed to represent the sensation must already be present in the idea designated by ‘this’. The same holds, Bolzano maintains, for any similar additions to the idea [this]:

¹ WL, §59, no. 3.

[A]s certain as it is that ideas of the form “This, which now occurs in me” are singular ideas, just so is it certain that among these there are at least some which are completely *simple*. For if we suppress the thought of any additions like “which occurs in me just now”, “which I just now see, hear, or feel”, “which I am now pointing at with my finger”, etc., the bare idea designated by the word “this” is certainly a completely simple idea. But the object that it represents remains throughout the same single one, whether we think the additions or not. For, if we consider them more closely, all these additions express no more than certain attributes which that single object which we just now represent possesses precisely because it is this one and no other; indeed, our idea does not become restricted to that single object only by means of these additions, but rather becomes redundant through them.¹

That is, the circumstances that it is *I* who am experiencing this pain, and that I am experiencing it *now*, follow from the circumstance that it is *this* and not something else.²

It might be thought that this last idea, despite the simplicity of the expression ‘this’, must nevertheless be complex. Perhaps it means something like: the sensation which I am presently experiencing? But then one is confronted with accounting for the ideas designated by the indexicals ‘I’ and ‘presently’. Given Bolzano’s analysis of the idea [I] as [that something which is conscious of certain ideas],³ we can see that the proposed description would lead us right back to [this] again. For in order to represent myself, I would have somehow to single out the ideas I am conscious of—and how to do this without referring to *this* or *that* idea?⁴ [Presently], for its part, is not specific enough in any case, for Bolzano would not accept that the mind can only experience one thing at a time. We would then have to say which of the present

¹ **ML**, §6, no. 4 [**BBGA** 2A.7, p. 59; **MM-EX**, p. 51]; cf. Bolzano to Exner, 9 July 1833 [**BBGA** 3.4/1, pp. 27 ff.; **MM-EX**, pp. 97–8]; **WL**, §278 [III.22].

² We shall discuss later (p. 317) what Bolzano might have meant when he said that the object of the intuition has these additional attributes follows from the fact that it is *this*.

³ **WL**, §44.2 [I.192].

⁴ Cf. “Verbesserungen und Zusätze zur Logik,” **BBGA**, 2A.12/2, p. 90: “On the idea ‘I’: I take this to be complex. By ‘I’, we mean the substance to which *these* ideas, sensations, etc., adhere, and this idea is composed of the concepts of a substance, of having, etc.—and the intuitions of *these* ideas, sensations, etc.”

experiences is *this* one—and again, it seems that we will not be able to do so without using indexicals, and thus running up against [this] once more—not to mention the fact that it seems impossible to fix a reference to a particular time without the help of intuitions. We will find the same thing happening no matter what description we attempt to substitute for ‘this’.¹ But when every attempt to define an idea turns out to be circular, we have good reason to believe it to be simple, as Bolzano claims elsewhere.² On the linguistic level, there is a demonstrative element here that cannot be eliminated, and its presence points to the existence of simple ideas with exactly one object, intuitions. Bolzano thus seems to have discovered something quite like Perry’s “essential indexicals” through his semantic analysis of judgments of perception. For it seems clear that any expression of such judgments will contain ineliminable demonstratives, indexicals, or like entities.³

Sensations, we have seen, can be the objects of intuitions. So too other mental occurrences such as ideas, judgments, and the like. But this is all, at least as far as humanly graspable intuitions are concerned.⁴ On Bolzano’s view, our immediate acquaintance with particulars is thus limited to the contents of our own minds. Since the objects of human intuitions are actual events that occur only once, there can be no second human intuition with the same object. Intuitions are as a consequence unrepeatable and *a fortiori* not

¹ Cf. B. Russell, *An Inquiry into Meaning and Truth* (many editions), Chapter 7: “Egocentric particulars”.

² WL, §350.

³ J. Perry, “The problem of the essential indexical,” in *The Problem of the Essential Indexical and other Essays* (Stanford: CSLI Publications, 2000). See M. Textor, *Bolzano’s Propositionalism* (Berlin: De Gruyter, 1996) for an examination of the connections between Bolzano’s theory and the modern semantics of indexicals and demonstratives. Bolzano’s intuitions are even more closely related to Russell’s *logically proper names* (see, e.g., *The Philosophy of Logical Atomism* [Reprint Chicago: Open Court, 1998]).

⁴ WL, §278 [III.22]; §286 [III.84–5]. At this point, it seems natural to ask whether Bolzano thought *God* had intuitions over and above those corresponding to human ones. Though we have found no detailed discussion of this question, a stray remark in the *Theory of Science* (WL, §81, no. 2 [I.390]) suggests that the answer is yes, and indeed that God has an intuition for *every* object: “I believe I may conclude that it must always be possible to determine the matter of an object by means of inner attributes alone because the constituents of which an object is composed belong to the object and can be thought by means of certain simple ideas that apply to these parts and nothing else, so that their determination does not require the consideration of another, different object (at least when we are speaking not just of our human cognitive faculties, but cognitive faculties in general).”

communicable. Intuitions are the only such elements in our ideas, however; all other simple or primitive ideas are both repeatable and communicable. This does not exclude the possibility of *mixed* incommunicable ideas however, those which incorporate intuitions along with publicly graspable concepts as parts. Among the most important ideas of this sort, Bolzano thinks, are those corresponding to proper names and some uses of natural kind terms like ‘gold’.¹ Representations of self will, as noted above, also be of this kind.²

(G) REPRESENTING INDIVIDUALS

Bolzano’s intuitions also allowed him to deal in a novel way with a problem that stubbornly resists solution in many theories of representation, the problem of explaining how it is possible to represent particular, contingently existing things. In order to indicate the general features of the problem we will sketch accounts due to Leibniz, Brentano, and Meinong before turning to Bolzano’s solution and the role intuitions play in it.

According to Leibniz’s essay on knowledge, truth, and ideas, the ultimate structure of an idea is that of a sum of simple (ideas of) characteristics.³ These characteristics, further, are supposed to be compatible in any finite combination, a view which entails that each finite idea may represent an infinite number of individuals. It follows from this that no finite idea can be known to have only a single contingently existing object—even if it so happens that only one individual in the universe has this combination of characteristics, we are never in a position to know this. Since only finite ideas can be dis-

¹ WL, §75, no. 4.

² Perry (“Frege on Demonstratives,” in *The Problem of the Essential Indexical and Other Essays* [Stanford: CSLI Publications, 2000], p. 15) has argued against incommunicable ideas of self as follows: “Suppose *M* is a private and incommunicable sense, which is to serve as the sense of “I” when I think about myself. *M* cannot be a complex sense, resulting from the compounding of simpler, generally accessible senses. For it seems clear that it is sufficient, to grasp the result of such compounding, that one grasp the senses compounded. So *M* will have to be, as Frege says, *primitive*.” His argument continues by pointing out the implausibility of this last mentioned view. But there is an unwarranted inference here: from the fact that an incommunicable complex sense must contain an incommunicable part it by no means follows that the incommunicable part must be an idea of self. [That which is thinking *this*] (or, as Russell would later have it, [the biography to which *this* belongs]), for example, would, on Bolzano’s account, be a complex, incommunicable idea of self with no incommunicable part representing the self.

³ G. W. Leibniz, *Philosophical Papers and Letters*, ed. and tr. L. Loemker, 2nd edn (Dordrecht: Reidel, 1989), pp. 291–5.

tinctly grasped, it follows that we can have no distinct ideas of individuals at all. Moreover, in Leibniz's way of thinking, every finite idea will represent infinitely many *possible objects*. In this sense, for the many philosophers who shared Leibniz's basic presuppositions but thought us incapable of thinking infinitely complex ideas, it follows that there can be no ideas of individuals at all. Our encounters with the things of this world would on such accounts take place through a fog of generality. For Leibniz, the thesis is rather an epistemological one: because of the infinite complexity of our ideas of individuals, we can never attain *clarity* with respect to them, i.e., can never be sure of reidentifying an individual we have once perceived.

You see, paradoxical as it may seem, it is impossible for us to know individuals or to find any way of precisely determining the individuality of any thing except by keeping hold of the thing itself. For any set of circumstances could recur, with tiny differences which we would not take in; and place and time, far from being determinants by themselves, must themselves be determined by the things they contain. The most important point in this is that individuality involves infinity, and only someone who is capable of grasping the infinite could know the principle of individuation of a given thing.¹

Various means have been tried to deal with this problem. Kant, for example, postulated a different species of idea (or representation), his intuitions, once he had convinced himself that concepts alone (Leibniz's representations) could not represent individuals.² Brentano, for his part, suggested that while we could not have ideas of individuals as such, we could nevertheless form what he calls an *indirect* idea of an individual, that is, an idea of an idea of an individual. This, he suggests, we can do as follows. Beginning with a general idea, we see that further determinations can be added to render it more specific. Thus, for example, beginning with [primate], we might add the determination [bipedal]; to [bipedal primate] the determination [with articulate speech], and so on. We understand that this process can continue *ad infinitum*, and that as a consequence, we can never have a completely determinate idea. Despite this, we can form an idea of an idea which is completely determinate, that is, such that no further determination could be added to it, just as, for example, one can form an idea of a real number as a theoretical limit to a certain kind of quantitative determination. In this way we arrive at an idea

¹ *New Essays*, III, iii §6, tr. Remnant and Bennett.

² See, e.g., *Logik*, ed. Jäsche, I, §11, §15.

of an idea of an individual.¹ Note, however, that this approach falls short in two ways. First, the idea that we have is a recipe for producing an idea rather than an idea itself and, what is more, a recipe that is admittedly impossible to carry out. Second, the shadow of an idea thus procured would only be of some possible individual, and we would neither know whether it picked out an existing individual, nor, even if it did, would we ever be in a position to know *which* one.

Meinong, in his treatise *On Possibility and Probability*, proposed a slightly different solution that met the first of these difficulties.² Admitting so-called “extra-constitutive” determinations like [being consistently and completely determined] as parts of ideas, he suggests that we can have an idea of an individual along the lines suggested by Brentano. In effect, we take any general idea and “fill it out” by adding to it the properties of being consistently and completely determined. Our idea is then not simply [primate], but rather [a completely determinate primate] or [a certain primate]. Note, however, that the second problem that dogged Brentano’s solution has not gone away. By virtue of representing a fully determinate primate, even an existing primate (for existence is also an extra-constitutive property), we will never be in a position to know which one we have thus represented. The problem which gave rise to Meinong’s speculations—how do I manage to represent *this* primate?—thus remains unsolved.

Nor will ideas of the circumstances of the act of representation help to resolve this difficulty. For if, as Brentano had maintained,³ ideas of time, space, and self are general in character, they cannot serve by themselves to fix a particular reference. And here Meinong’s trick of filling out is of no use—for filling out a self-representation, for example, can take place in an infinity of different ways, and so there is no way of knowing on this account that [a determinate primate in front of a determinate me] represents the primate in front of me, and indeed the probability that it does is infinitesimally small.

Bolzano’s solution to the problem respects the presuppositions of these philosophers. He agrees that even were it to be the case that there was only

¹ Brentano, dictation 9 March 1917 (in F. Brentano, *Psychology from an Empirical Standpoint*, tr. L. McAlister, p. 320).

² *Über Möglichkeit und Wahrscheinlichkeit*, ed. R. Chisholm, in A. Meinong, *Gesamtausgabe* (Graz: Akademische Druck- u. Verlagsanstalt, 1968–78), Vol. 6, Chapter 2.

³ *Psychology from an empirical standpoint III: sensory and noetic consciousness*, tr. L. McAlister (London: Routledge & Kegan Paul, 1981), pp. 66–7. This point can be made in a different way by pointing out the indexical character of the words ‘I’, ‘now’, ‘here’, etc.

one existing individual with a certain (finite) combination of conceptually represented characteristics, we would never be in a position to know this.

It is indeed false that there are even as many as two real things that are completely equal to one another in all their (inner) attributes. One might hope that a number of these inner attributes, each of which can be comprehended by a pure concept, could be used to form a concept that fits only this and no other object; but even if we suppose that for each object there is a finite number of inner attributes which no other object has in just this combination, all of which can be represented by pure concepts, it is clear, nonetheless, that we can never know whether the attributes that we have combined in our concept are really of such a nature. From the fact that we do not know a second object that has all these properties it does not follow that no such object exists in some unexplored region of the universe.¹

Because of this, not even concepts that by their form are singular (like those corresponding to definite descriptions involving only terms designating pure concepts) can be known to represent actually existing individuals.²

Intuitions, by contrast, do represent single objects and can be known to do so. Also, because the objects represented by intuitions have actual existence and thus causal connections with other real things, it is possible to form mixed ideas (i.e., ideas containing both concepts and intuitions as parts) which refer to individuals as the causes of certain intuitions we have had:

...if our intuitions are caused by the influence upon our senses of some external object, we may wish to indicate this object to others. If it is an enduring object, and if it recurs and is of sufficient importance, then a special sign is formed for it: a *proper name*. Thus, proper names always designate mixed ideas of the form “the object that is the cause of my once having had such and such intuitions.”³

He continues:

This holds not only of proper names designating external objects that influence our own senses, but also of proper names designating an object that has long since ceased to act upon our senses,

¹ WL, §74, no. 3 [I.333–4].

² WL, §76.

³ WL, §75, no. 2 [I.335].

e.g., Socrates. To the question what kinds of intuitions our idea contains in such a case, I would answer that, e.g., by “Socrates” we mean a philosopher, “who lived so and so many centuries ago in Greece and who was called Socrates”. If nowhere else, there are intuitions at least in the sounds of which the name “Socrates” is composed.¹

Thus our representations of existing individuals (almost) always involve intuitions (God is an exception, since He can be represented by a pure concept, e.g., [unconditioned actuality]).

As it stands, Bolzano’s account is a mere sketch, and needs some work. To begin with, he maintained that the immediate causes of subjective human intuitions were always mental events, e.g., sensations. But clearly a sensation was not the object he had in mind in claiming the meaning of proper names such as ‘Socrates’ to be indicated more distinctly by descriptions of the form ‘the object that is the cause of my once having had such and such intuitions.’ In light of this, it seems reasonable to interpret his characterization of proper names either as adverting to the *mediate* cause of my forming certain intuitions (i.e., as the cause of my having certain sensations), or else as mistakenly speaking of intuitions rather than the *objects* (sensations) these represent.² Even so, problems would remain. For Bolzano also held that every finite substance contributed causally to the formation of any given sensation. For a sensation is an actual entity, and of these he writes:

[T]he complete cause of the existence and the attributes of [a single fully determinate actual object] can only be the totality of

¹ **WL**, §75 [I.335]. Bolzano’s account closely parallels the one later developed by Bertrand Russell. See, e.g., *The Problems of Philosophy*, Chapter 5; also “The Philosophy of Logical Atomism,” Part II; quoted from B. Russell, *Logic and Knowledge*, ed. R. C. Marsh (London: George Allen & Unwin, 1956). In another striking coincidence, Russell also appeals to the perception of utterances of proper names to account for reference to persons no longer living: “[W]hen, for example, we make a statement about Julius Caesar, it is plain that Julius Caesar himself is not before our minds, since we are not acquainted with him. We have in mind some *description* of Julius Caesar: ‘the man who was assassinated on the Ides of March’, ‘the founder of the Roman Empire’, or, perhaps, merely ‘the man whose name was *Julius Caesar*’. (In this last description, *Julius Caesar* is a noise or shape with which we are acquainted.)” *The Problems of Philosophy*, new edn (Oxford: Oxford University Press, 1997), Chapter 5, pp. 58–9.

² In his earlier work **BD**, Bolzano uses the term ‘intuition’ where his later usage would require ‘sensation’ or ‘impression’, etc. See, e.g., §6 of the Appendix. It is possible that the habit of such usage survived the change of doctrine.

all existing objects. For each of them has some influence upon it, no matter how small, so if that other thing did not exist, our object would be different.¹

Thus he owes us an account of how one particular object is singled out as the referent of a proper name. He likely thought we had the ability to identify a sub-collection of the infinite multitude of substances as primarily responsible for our having certain sensations. Exactly how this is supposed to be accomplished, however, is not precisely indicated.²

A final point seems worth mentioning here. Because proper names designate ideas with intuitions among their parts, no two individuals will associate the same idea with a given proper name. Indeed, since intuitions are unrepeatable, a single individual will produce unequal ideas each time he understands a proper name. Thus the meaning of names, strictly speaking, is incommunicable and unrepeatable. While recognizing this consequence of his theory, Bolzano nonetheless maintains that we can still speak of communication in such cases, provided that we accept a weaker notion that only requires extensional equivalence:

[I]t is certainly well-founded to say that a single empirical truth grasped by one person can never be completely grasped by a second person, nor indeed by the same person at another time. Strictly speaking, it is no longer the same proposition that I put forward when I now say that Alexander was born roughly 2190 years ago (before *this* instant) as when I said this an hour ago. For the intuition of that which I presently feel or think that lies in the words “before *this* instant” is now different from that of an hour ago. Similarly for the propositions “Sirius is a fixed star,” inasmuch as the intuitions found in the idea “Sirius” are different for every man. But these differences will be seen not to matter here, and so can be disregarded. To say that we have communicated the truths contained in these propositions to the reader, it suffices that he form ideas that, if not composed of the same intuitions as ours, are nevertheless equivalent to them, i.e., that he thinks of the same object (the same subject), and attributes the same property (the same predicate) to it that we do. Thus in the first example it will be sufficient if our reader connects an idea with the word “Alexander” that in fact refers only to this

¹ WL, §379 [III.501].

² Some related problems are discussed in WL, §303.

Macedonian king, and learns that this man was born 2190 years before the year this was written. Thus understood, there can be no doubt that empirical truths can also be communicated through writing...¹

Whether Bolzano provides an adequate account of the semantics of proper names may well be doubted (although it might provide an adequate account of singular reference in certain respects); that is not our concern here, nor, obviously, was it his main preoccupation. Let us simply point out that with his intuitions, and without violating the spirit of the traditional psychology of perception, he managed to provide at least a partial explanation of our ability to represent existing individuals, something which eluded a good many thinkers before and after him.

(H) REPRESENTING NATURAL KINDS

The problem of representing natural kinds is closely related to that of representing individuals, and causes similar sorts of difficulties for traditional theories of representation.² Some of the most prominent examples of natural kinds are biological: here we have not only resemblance between members of one and the same species, but also the fact that species breed true (lions produce lions, crows produce crows, lilies produce lilies, and so on). Other examples are commonly occurring elements (e.g., silver, gold, iron) or compounds (e.g., water, oxygen, glucose).

It might be thought that we pick out natural kinds by means of qualitative descriptions, or complex ideas of observable properties. Such was, for example, the view of Locke.³ On this view, we would consider a sample of naturally occurring things, compare their observable properties, and perhaps form a complex idea consisting of ideas of some of the properties the individuals in our sample have in common. This idea, which Locke calls a *nominal essence*, then determines a *species* or *sort*, consisting of precisely those things that have these properties. According to Locke, the nominal essence of gold, for example, “is that complex *Idea* the word *Gold* stands for, let it be, for instance, a Body yellow, of a certain weight, malleable, fusible and fixed.”⁴ At the same time, Locke claims that natural species have what he calls a *real*

¹ WL, §410, Note [IV.48–9].

² Indeed, on Leibniz’s view, the two problems are at bottom the same, since every individual constitutes its own species, or natural kind.

³ *Essay*, III, vi.

⁴ *Essay*, III, vi, §2.

essence, “that real Constitution of Substances, upon which depends this *nominal essence*.”¹ The real essence of gold, for example, is “the constitution of the insensible parts of that Body, on which those qualities, and the other properties of *Gold* depend.”² The real essence, Locke continues, is in most cases unknown to us and destined to remain so, so there can be no question of the real essence being reflected in the meaning of a word such as ‘gold’, nor in our thoughts about natural substances.³

Locke’s account is open to a number of objections, which can be illustrated through consideration of a couple of examples. Suppose, that a naturalist has collected a variety of insects, and subsequently sorted them into different groups based upon their observable characteristics. Let us also suppose that he has done the sorting in such a way that the individuals in each group do in fact all belong to the same biological species. Following Locke’s procedure, he then forms a nominal essence of one of these species, consisting of ideas of some of the observable properties of the insects in the sample.

From the point of view of capturing a natural kind (the species) with this idea, he may fail in several ways. First, his idea may be too broad, covering not only members of the natural species to which the insects in his sample belong, but others as well. This is a constant danger with insects, precisely because mimicry is so common among them. There are wasps that look like ants, flies that look like bees, beetles that look like bugs, different species of butterflies that look like each other, and so on. Putting together a simple list of observable characteristics can thus net several species quite easily.

Second, his idea may be too narrow. If, for instance, his sample consists entirely of adult insects, he may easily form an idea that does not fit the larval or nymphal forms, etc. Or perhaps he has a sample of aphids where all the insects are wingless females that reproduce parthenogenetically. If he uses these characteristics to form a nominal essence, he will exclude insects descended from these (and thus of the same natural species) which have wings and reproduce sexually.

Third, the idea may be incorrect, in that the idea he forms may even incorporate characteristics that do not belong to the objects in his sample. In this case, he forms a nominal essence that does not even cover his sample. This might be the case, e.g., if he miscounted the number of segments in the antennae, and used the incorrect number in forming the nominal essence for the species.

¹ *Essay*, III, vi, §2.

² *Essay*, III, vi, §2.

³ *Essay*, III, vi, §6.

In all three cases of failure, the naturalist would not have an idea of the natural kind, i.e., the species to which the members of the sample belong, at all. Interestingly, it follows from this that he could not make incorrect judgments about that species—for in order to make an incorrect judgment *about X*, he needs an idea *of X*, and that is precisely what is lacking in these cases. In the first case described above, for example, if the naturalist judges that all the members of the (nominal) species have one of the characteristics enumerated in the nominal essence, his judgment is entirely correct and indeed analytic in Kant's sense, even if some members of the biological species from which his sample was drawn lack that characteristic. In the second case, similarly, he would not be incorrect in thinking that all the members of the (nominal) species picked out by his idea are wingless females, since that judgment, too, would be analytic in Kant's sense. Finally, in the third case, if he believes the insects in the species he is thinking about have, say, thirteen antennal segments, he is quite correct, even if the insects in his sample only have twelve.

There are further problems. Failure to form adequate nominal essences is to be expected, especially in the early stages of research. But if we lack an idea of the species in question, we are in no position to inquire about its real essence. And if two naturalists, working independently, have each collected samples of the same species of aphid, and formed inadequate nominal essences with different extensions, then they cannot even be said to disagree concerning the characteristics of the species their samples belong to, since neither is in fact talking about that species, according to Locke. Both may be perfectly correct in their assertions even where these are apparently incompatible, one saying, for instance, "Aphids have wings" and the other denying this. Locke's account thus threatens to make scientific inquiry unintelligible.

Bolzano saw that his intuitions could be put to use in explaining how we manage to represent natural kinds.¹ The main problem with Locke's account is that there is no reference at all to the sample. Instead, once the naturalist has discerned a handful of characteristics of the members of the sample, it is simply disregarded, and the complex of characteristics becomes the sole focus, with the attendant problems we enumerated above. But this sample is his only link to the species. In Bolzano's approach, by contrast, the sample continues to play a key role in representing the species. Even if we cannot pick it out in purely conceptual (descriptive) terms, we may nonetheless represent it as it were *demonstratively*, as [*this* species] or, more precisely, [the species to which *these* objects (or organisms, etc.) belong].

¹ WL, §75, no. 4.

For Bolzano, as we saw above, the source of the “demonstrative” element in representation is intuitions, i.e., simple, singular ideas. Since these always represent mental states or events in Bolzano’s view, ideas of natural kinds must somehow be formed with the help of these. The key insight is that intuitions representing sensations caused by the objects in a sample can be parlayed into an idea that picks out the natural species to which the sample belongs. Such an idea might be, e.g., [the species to which the objects that caused *these* sensations belong] or, more simply, [the species to which the objects that caused *this* and *that*, etc., belong]. Writing of gold, for example, he says this:

[N]atural scientists are not at all averse to applying such names [sc., “gold”, “silver”, “oxygen”] to any substance in the universe so long as it has the same inner attributes that this substance has on Earth. However, we know only very few of the inner attributes of these substances other than through their influence upon further substances and finally upon our own senses and ourselves (i.e., our sensory and representative faculties). Thus we only know them through relations which they have to certain objects that are given through *intuition* alone. Consequently we tend to express these attributes in terms of relations and describe, e.g., gold as a body that causes the idea of a yellow colour in our organ of sight, thin layers of which transmit green light, that is 19 times heavier than water, etc.¹

Of course, failure of reference (or representation) can still occur in such cases—for example, if the sample contains objects belonging to several rather than to a single species. But Bolzano’s account, unlike Locke’s, also accommodates the possibility of success. Moreover, it is possible, with the help of such *mixed* ideas, to represent *indirectly* a pure concept representing what Locke called the real essence of a natural species. Bolzano in fact thinks that natural kind terms are sometimes used to designate such pure concepts:

We can take the effects of gold upon our sight and upon other objects of intuition merely as signs of certain attributes of gold that can be determined through pure concepts. For instance, we can think of the words that “it looks yellow to us” as being nothing but an expression for a certain inner attribute of gold which is the reason why gold produces the idea “yellow” in an organ such

¹ WL, §75, no. 4 [338–9].

as our eye, etc. If we understand matters in this way, then the intuitions which occur in the verbal expression of our concept of gold cannot belong to the content of this concept, but must belong merely to the means by which we *designate* those unknown inner attributes of gold that can only be determined through pure concepts, and of whose ideas the concept of gold is to be composed.¹

In such cases, he observes:

... we can say that the concept designated by the word “gold” is a pure concept, but one that is not completely known to us, analogous to the value x in an equation we haven’t solved.²

Finally, Bolzano thought that natural kinds terms were often used *ambiguously*, sometimes designating pure concepts, sometimes mixed ideas, and that we sometimes switch from one meaning to another without noticing it.³ As an example, he points to the term ‘human’, which, he claims, may designate either the idea of a being with a rational mind and an organic body (a pure concept) or else the idea of a being of this sort that is found *on Earth* (a mixed concept, given the intuitions contained in the idea [Earth]). Bolzano recognized that such ideas could have different extensions (like many of his contemporaries, he was quite confident that other planets were inhabited by rational beings). It seems clear, too, they could give rise to different truth-values for modal as well as non-modal claims.

Though the similarities should not be exaggerated, Bolzano’s approach to ideas of natural kinds bears some resemblance to the influential account developed by Hilary Putnam.⁴ In particular, we think it fair to say that Bolzano recognized the indexical character of at least some natural kind terms.

(I) CONCLUSION

Despite its deficiencies, Bolzano’s account of ideas constitutes a significant and radical break with dominant currents of early modern philosophy. Ideas are no longer looked upon as stand-ins for things, and any visual associations

¹ WL, §75, no. 4 [I.339].

² WL, §75, no. 4 [I.340].

³ WL, §75, no. 4. He credits Locke with this observation (*Essay*, IV, iv).

⁴ See, e.g., H. Putnam, “Meaning and Reference,” *Journal of Philosophy* 70 (1973) 699–711; “The Meaning of ‘Meaning’,” *Minnesota Studies in the Philosophy of Science* 7 (1975) 215–71.

of the terms *Vorstellung* and even *Anschauung* are flatly rejected. Instead, we find something very like a theory of reference at the heart of Bolzano's account of ideas, the difference being that he speaks not of linguistic expressions but rather of what Frege would later call the *senses* of these expressions. For Bolzano, we know what an idea is when we know if it is simple or complex and, if complex, what its parts are and how they are combined. In the case of ideas with objects, we know what a simple idea is when we know its extension, and that's all there is to say. It is instructive to compare this account with the way that semantics is usually treated today by logicians, where a formal language is interpreted precisely by assigning extensions to the basic non-logical vocabulary (i.e., the non-logical constants). Bolzano's account is, in its essentials, the view of contemporary logic, and it would be difficult to overestimate the gulf separating him from previous logicians, or indeed from many later ones.

6. CONTRIBUTIONS TO THE ANALYSIS OF PROPOSITIONS

As noted above, Bolzano did not believe that he was in a position to provide a definitive account of the possible forms of propositions. He nevertheless had a good deal to say on the subject. A large portion of Volume Two of the *Theory of Science* is devoted to the analysis of propositions, and the interpretation of their verbal expressions. Among these are statements involving conditionals, disjunctions, negation, necessity, possibility, equality, identity, existence, universal quantification, etc.

His goal was to find more distinct expressions for a variety of kinds of propositions. A sentence would be a perfectly distinct expression of a given proposition in itself if each of its elements corresponded to a simple idea in the proposition, and the elements of the sentence were connected in the same way as the elements of the proposition. Expressions can fall short of such perfection in various ways: they may, for example, contain too few or too many elements, or forms of combination that are not found in the proposition in itself, or they may use a simple expression to designate something that is logically complex. Among such imperfect expressions, however, some will be more distinct than others, displaying more relevant detail, even though they may still fall short of perfect distinctness.

The goal of discovering perfectly distinct expressions of propositions might seem to require no further explanation: a proposition in itself is what it is, it has the components and structure that it does, and a sentence either expresses them distinctly or it does not. It seems to us, however, that

such an explanation could not be completely satisfactory to Bolzano. For, to begin with, he thought that there was no interaction between propositions in themselves and our thought. There can, therefore, be no direct comparison between a proposition in itself and a linguistic expression. The best we seem to be able to do is compare thoughts and linguistic expressions among themselves, and, based upon such considerations, decide that some expressions are more distinct than others.

Even here, Bolzano does not always seem to aim at strict synonymy, i.e., identity of sense, but contents himself on many occasions with getting the truth conditions right. Speaking of his analysis of propositions of the form ‘Some *A* are *B*’, for instance, he writes:

It might be objected that it is unlikely that my explication and the linguistic expression which it explicates both indicate the same thought, since they are composed of entirely different words. I do not wish to quarrel about this matter; we only need to admit that the sense connected with the original formulation is equivalent to the sense of my explication, i.e., that whenever one of them is true the other is too.¹

Though he does not discuss these matters as thoroughly as one might have wished, it would seem that one important constraint on the analysis of propositions is provided by the theory of inferences.² Our inferential practice is guided by linguistic forms, and these may often lead us astray, producing correct results in some cases but incorrect ones in others. That is, while two sentences may have the same truth-conditions, one may be more likely than the other to lead us into mistaken inferences. In some books, for instance, it was said that ‘Some *A*’ designated the subject-idea of the proposition [Some *A* are *B*]. Accordingly, [Some men] would be the subject-idea of [Some men are virtuous.] At the same time, if we adopt Aristotle’s definition of truth (and Bolzano, like Tarski after him, thought that Aristotle’s definition was not half bad³), an affirmative sentence is true just in case it *says* that a certain object

¹ **WL**, §137 [II.53–4].

² See **WL**, §366 [III.449–50], quoted above, p. 200. Another constraint discussed by Bolzano is non-redundancy. In many languages, for example, *number* is marked by both verb and subject forms, e.g., ‘He has’, ‘they have’ (**WL**, §127 [II.15]). Arguably, however, there is really only one contribution to meaning here, even though it is expressed twice. Languages may also contain markers that at least on the face of things often express nothing of real significance, e.g., gender (cf. **WL**, §57.3; §69, note 2; §127, no. 4; also Mark Twain, “The Awful German Language”).

³ Cf. below, p. 250 *et seq.*

has a certain property, and the object *has* the property in question. But the idea [some men] would appear to represent, say, Hitler, Caligula, Stalin, and Pol Pot. The above interpretation would thus seem to commit us to saying that these men were virtuous. Bolzano's proposal that we take [Some men are virtuous] to be more distinctly expressed by 'The idea of a man who is virtuous has an object' does not have this absurd consequence.¹ Or consider the simple sentence 'It is snowing.' Bolzano, along with traditional logic, held that if a subject–predicate proposition [*A is B*] is true, then so too is the proposition [There is an *A*.] On a straightforward grammatical analysis, with this in mind, we might be tempted (as some philosophers were²) to infer that 'it' refers to an object. By contrast, the sentence 'The idea of a snowfall in this place at the present moment has an object', although admittedly clunky, will not tempt us to hunt for the elusive referent of 'it'.³

(A) ON THE FORM AND CONTENT OF PROPOSITIONS

Kant, as we mentioned at the beginning of this chapter, may be credited with stimulating a logic-revival among German-speaking philosophers, and his influence was impossible to miss in the German logics of Bolzano's day. It was evident in the conception logicians formed of their subject as a *purely formal* science, and especially so in the adoption of Kant's table of judgments as a definitive account of their possible forms.⁴ Logic, the followers of Kant repeated tirelessly, concerns itself only with the form of thought, disregarding any differences having to do with content.⁵ This focus on form was thought to explain both the great success of logic and its limitations:

That logic has been so successful in following the secure path of a science is an advantage that it owes entirely to its limitations. They entitle it, even obligate it, to abstract from all objects of cognition and their differences; hence in logic the understanding deals with nothing more than itself and its form.⁶

¹ Bolzano discusses this example in §173 of the **WL**.

² Some of these attempts are documented in Anton Marty's classic "Über subjectlose Sätze und das Verhältniss der Grammatik zu Logik und Psychologie," *Vierteljahrsschrift für wissenschaftliche Philosophie* 8 (1884) 56–94, 161–92, 292–340.

³ Cf. **WL**, §172.

⁴ Well, nearly definitive. A number of Kantian logicians suggested minor improvements to Kant's table.

⁵ See, e.g., Kant, *Critique of Pure Reason*, A54/B78.

⁶ *Critique of Pure Reason*, B ix.

Because the form of thought is contributed by the mind, it was claimed, it can be known *a priori*, and constitutes *necessary* features of thought, which explains the apriority and necessity of logic. At the same time, there wasn't much to logic, for the possible forms of thought were assumed to be readily embraceable in a survey, and not at all hard to grasp, accounting for the complete state of logic since the time of Aristotle.¹

Bolzano was sceptical of the Kantian claims about logical form from the start, and his survey of the literature only confirmed his doubts. If form were so well-understood, and the distinction between form and content (or matter) so important, it must be possible to give a precise general definition of form. Yet those offered by Kant and his followers were far from adequate. And the arguments offered in support of the definitiveness of Kant's table were far from convincing.²

Attempting to introduce some order into his discussion of these matters, Bolzano begins in §81 of the *Wissenschaftslehre* by distinguishing several senses of the terms 'form' and 'matter'. In many cases, he remarks, 'form' refers to the way the parts of a complex whole are combined. Thus, for example, a pile of building materials would differ from a house constructed with them in respect of form.

In this sense, the ideas:

[A learned son of an ignorant father] / [An ignorant son of a learned father]

as well as the propositions:

[Some politicians are not men] / [Some men are not politicians]

although containing the same constituents, would differ in the arrangement of these constituents, and thus in terms of form.

If logic is sometimes called a purely formal science, it had better not be in this sense, Bolzano claims, for, to consider but one example, the concepts [not] and [some], as they occur in the two propositions above, would then fall outside the scope of logic, which is surely not what the Kantians intended.³

¹ Cf. *Critique of Pure Reason*, A xiv: "...common logic already provides me with an example [which shows] that all simple acts of reason can be enumerated completely and systematically."

² Bolzano reviews the relevant literature in §§185–94 of the **WL**. See also §§12, 81, 116, and 254.

³ **WL**, §116 [I.540].

In a second sense, Bolzano claims, ‘form’ is taken to be equivalent to ‘kind’ or ‘species’.¹ In a way, logic can be said to be a formal science in this sense, since, generally speaking, it formulates theorems not about individual propositions, ideas, inferences, etc., but rather about entire classes of them:

It is the task of logic to give rules which apply simultaneously to several truths or, what amounts to the same, to a whole collection of truths. For this reason, the theorems (though perhaps not the examples) of logic never concern a particular, fully determinate proposition, i.e., a proposition in which subject, copula and predicate are all given. Rather, theorems concern a whole collection of propositions at once, i.e., propositions some of whose parts are determined, while the remainder is undetermined. Thus the proposition “Some people have white skin” occurs in logic at best as an example, and not as the subject of a theorem, while a collection of propositions, such as that determined by the expression “Some *A* are *B*” may well be the subject of a theorem. If these collections are to be called general *forms* of propositions, then it is permissible to say that logic is concerned with forms rather than with individual propositions.²

Logicians often deal with forms by means of linguistic expressions. Thus beginning with a sentence such as ‘Some men are wise’, we can replace ‘men’ and ‘wise’ with letters to produce the expression

Some *A* are *B*

—a linguistic form which, along with a stipulation concerning which substitutions for *A* and *B* are permissible, determines a class of sentences including

Some dogs are vicious, Some numbers are perfect, etc.

To the extent that the structure of these linguistic objects reflects that of the propositions they express, one can also say that the form ‘Some *A* are *B*’ determines not only a class of sentences, but also a class of propositions. This observation motivates a third sense of ‘form’, according to which ‘form’ refers to a linguistic expression that determines a kind of propositions (or ideas, arguments, etc.):

¹ In support of this somewhat unorthodox view, Bolzano notes (WL, §81, note 1 [I.391]) that Cicero took the terms *forma* and *species* to be interchangeable.

² WL, §12, no. 2 [I.48].

When I speak of the ideas, propositions and arguments that stand under a certain *form*, then I mean by 'form' a certain concatenation of words or signs in general, which can represent a certain kind of idea, proposition, or argument. Thus, if the letter *A* represents any subject idea, and the letter *b* any attribute-idea, then the expression '*A* has *b*' is the general form of any proposition whatever, since all propositions can be represented by this concatenation of signs.¹

Note that since we can always choose to consider a different selection of constituents variable, it makes no sense in general to speak of a unique form of, say, a proposition or an argument. Moreover, form has a pragmatic aspect: what counts as form depends upon what general features we happen to be interested in:

In my opinion, what belongs to the *form* of a logical object are those of its attributes which determine the kind of object the logician intends to consider it to be.²

Consider, for example, the proposition:

[Aristotle is wise and Achilles is vain.]

By considering various parts of this proposition variable, we could arrive at a number of different forms, among them:

X is wise and Achilles is vain.	Aristotle is wise * Achilles is vain.
X is P and Achilles is vain.	X is P * Y is Q.
X is P and Y is vain.	A and B.
X is P and Y is Q.	A * B.

where X, Y mark places for singular ideas, P, Q for general ideas, A, B for propositions and * for a binary propositional connector.

In more complex propositions, one might distinguish hundreds or even thousands of forms. Since there is no upper bound on the complexity of propositions, it seems clear that the number of propositional forms is infinite.³ Not only are these forms not in us in the way the Kantians had supposed, there will also be a great many that can never be in us on account

¹ WL, §81 note 2 [I.393]; cf §12 [I.48].

² WL, §254 [II.516].

³ Cf. WL, §186 [II.252].

of their complexity and the limitations of our cognitive abilities. We do not impose these forms upon the objects of our thought—we take this to be one of the central thrusts of Bolzano’s claim that propositions and ideas in themselves are prior to thought—it would be more accurate to say that they impose themselves on us.

(B) THE SUBJECT–PREDICATE FORM

Although every proposition belongs to several forms, Bolzano argues in §127 of the *Theory of Science* that there is one form that all of them belong to, namely, the subject–predicate form ‘*A has b*’. In the case of a true proposition, [*A*], the subject-idea, will represent one or more objects, and [*b*], the predicate-idea, one or more attributes; false propositions may or may not have objectual subject- or predicate-ideas.¹ He states that the copula should not be expressed by the verb ‘to be’, since this may misleadingly suggest that the subject *exists* (for ‘to be’ is certainly used on some occasions to express existence, as in: I think, therefore I *am*.) The concept of predication is clearly different from the concept of existence, and if we keep the verb ‘to be’ for the latter, we should find another for the former.

Bolzano suggests ‘to have’ as a better choice, where the verb is meant to express the possession of an attribute.² Instead of ‘Socrates is mortal’, for example, he prefers ‘Socrates has mortality.’ He defends the view that the concept of having is the copula in all propositions, and expends more than a little ingenuity in arguing that neither negation nor time determinations (e.g., as marked by tense) nor modifiers such as ‘possibly’, ‘probably’, ‘certainly’ (as in ‘Sam is *probably* going to be late’) should be interpreted as belonging to the copula.

Elsewhere, Bolzano concedes that he does not have a fully convincing argument for his claim:

The only way B. knows to prove it is through an incomplete *induction*, ... attempting to show that any sentence that occurs to

¹ WL, §127 also contains one of the oddest assertions Bolzano ever made, namely: “Could we not call every compound of the form ‘*A has b*’ a proposition, no matter whether the signs ‘*A*’ and ‘*b*’ designate mere ideas, or ideas of a certain sort, or even entire propositions?” If we were to take this suggestion at face value, however, then it seems we would have to accept that ‘has has has’, ‘has has Socrates’, ‘is larger than has Socrates is mortal’, ‘perpendicularity has Aristotle’, etc., express propositions, have truth values, etc. One of Homer’s nods? (We find a different view in §81, note 2: there he adds that *A* must be a subject-idea and *b* an attribute-idea.)

² Cf. *Bolzano’s Wissenschaftslehre und Religionswissenschaft in einer beurtheilenden Uebersicht* (Sulzbach, 1841), p. 49.

him which appears to have a different form can be reduced to an expression of the form: *A has b*. . . . We cannot say whether readers will be completely satisfied with these reductions; the only thing that is certain is that even in their entirety they cannot be looked upon as a complete proof.¹

It was a commonplace of early analytic philosophy that the subject–predicate form was the bane of traditional logic and philosophy, and that only with the advent of modern symbolic logic did it become possible to arrive at an adequate philosophical grammar. Bolzano’s work on the analysis of propositions casts some doubt on such claims. It is sometimes said, for instance, that a subject–predicate logic can at best capture the argument forms and inferences of monadic first-order predicate logic. We will see that this is not so with Bolzano’s subject–predicate form—for, in addition to the basic subject–predicate form, much depends upon the view taken of the possible sorts of subjects and predicates.² At the same time, we shall also have occasion to observe various drawbacks of Bolzano’s position, notably, the contortions involved in paraphrasing some sentences into the prescribed form, contortions which, among other things, appear to open the door to some well-known antinomies.³

(C) TRUTH

Having convinced himself that all propositions belong to the form ‘*S has p*’, Bolzano thought he was in a position to give a fairly straightforward definition of truth. He begins by distinguishing a number of senses of the words ‘true’ and ‘truth’, namely:

1. Truth as an attribute of propositions in themselves, “by virtue of which they state something to be as it *is*.”
2. Truth as a proposition in itself possessing this attribute.
3. Truth as a judgment whose matter is a truth in sense no. 2

¹ Bolzano’s *Wissenschaftslehre und Religionswissenschaft*, p. 48.

² In this connection, it is worth recalling that Frege (*Begriffsschrift*, §3) had said that all of the formulas of his concept script could also be taken to be of the subject–predicate form, namely, by taking the combined content- and judgment-stroke ‘ $\vdash$ ’ to express the predicate in every case.

³ See Peter Simons, “Bolzano über Wahrheit,” pp. 13–27 in E. Morscher, ed., *Bernard Bolzanos geistiges Erbe für das 21. Jahrhundert*, Beiträge zur Bolzano-Forschung, Vol. 11 (St Augustin: Academia Verlag, 1999).

4. Truth as a collection of truths in either of the senses no. 2 or 3.
5. ‘True’ taken to be roughly synonymous with ‘genuine’—as in “a true work of art.”

Sense no. 1 is taken to be primary, nos. 2–4 derivative, and no. 5 improper; accordingly, he fixes his attention on the first one.¹

The rough characterization we find in no. 1 is refined a few pages later in §28: A proposition [*S* has *p*] is true, we read there, if and only if *p* belongs to *S*. Since to say that *p* belongs to *S* is just to say that *S* has *p*, however, we end up with the following:

The proposition [*S* has *p*] has truth iff *S* has *p*.

For example:

[31 has primality] has truth iff 31 has primality.

When the subject-idea is general, things become more complicated. Consider, for example, the proposition:

[Man has mortality.]

On Bolzano’s understanding, the subject-idea [man] is taken here in its full extension, but distributively, i.e., the sentence ‘Man has mortality’ is taken to express the same proposition as ‘Each man has mortality.’ One consequence of this is that the propositional forms ‘*X* has *y*’ and ‘*X* has non-*y*’ do not represent contradictories, even if we require the subject-idea to be objectual, for both may be false. “This happens when we replace *X* with an idea which applies to several objects, and by *y* an attribute-idea which holds of some, but not of all.”²

Predicate ideas, too, are often general in Bolzano’s opinion. But when this occurs, they are *not* taken in their full extension:

¹ WL, §24. As an historical curiosity, we mention that Heidegger seems to have made it at least this far into the *Theory of Science*, for he gave the same list of senses of ‘true’ and ‘truth’, using several of Bolzano’s phrases, in a course of lectures on logic given in 1925–6. See W. Biemel, ed., *Logik: die Frage nach der Wahrheit*, M. Heidegger, *Gesamtausgabe*, Vol. 21 (Frankfurt am Main: Klostermann, 1975), pp. 9–10. Bolzano is not mentioned as the source in the text, though he is briefly discussed later on (see, e.g., pp. 86–7).

² WL, §159, no. 29 [II.159].

For when, for example, we set out the judgement “Caius has intelligence” we do not wish to say that Caius has every kind of intelligence there is, for example, a well-developed, as well as a crude, a human, and an angelic, intelligence, etc. Thus an idea which occurs as the predicative part of a proposition is in no way taken in its full extension; rather we must say that the proposition leaves entirely undetermined which of the many attributes that stand under the predicative part (in case there are more than one) belong to the one object, or to the several objects, that stand under the idea *A*.¹

Thus a proposition [*A* has *b*] is equivalent to the following:

[E]very object standing under *A* has one of the attributes that stand under *b*; and if there are several of the latter, it remains undetermined which of them belongs to each *A*.²

Finally, since Bolzano took universal claims to have existential import, we have the following: [*S* has *p*] is true iff:

1. There is an object that stands under the idea [*S*]; and
2. Every object that stands under the idea [*S*] has an attribute that stands under the idea [*p*].

Bolzano assumes that any proposition that is not true is false. It follows that propositions with objectless subject-ideas, as well as those with objectless predicate-ideas, are false.

As proper method requires, Bolzano tries to show that the concept of truth is objectual, i.e., that there is at least one proposition with this property. His argument, which is adapted from one he found in Aristotle and Sextus Empiricus, runs as follows:

¹ **WL**, §131 [II.26].

² **WL**, §131 [II.26–7]. For similar formulations, see Bolzano’s letter to Exner of 18 December 1834 [**BGGA** 3.4/1, p. 111; **MM-EX**, p. 167]; “Verbesserungen und Zusätze zur Logik,” **BGGA** 2A12, p. 105. Bolzano appears to make a stronger claim in **WL** §131, namely, that the two sentences have the same sense. But this seems excessive, since the latter seems to express a proposition containing ideas of ideas, while this will not be the case generally with [*A* has *b*]. On this point, cf. E. Casari, “An interpretation of some ontological and semantical notions in Bolzano’s logic,” pp. 55–105 in *Bolzano’s Wissenschaftslehre 1837–1987: International Workshop Firenze 16–19 Sept. 1987* (Florence: Leo Olschki, 1992), p. 77.

That no proposition has truth disproves itself because it is itself a proposition and we should have to call it false in order to call it true. For, if all propositions were false, then this proposition itself, namely, that all propositions are false, would be false. Thus, not all propositions are false, but there are also true propositions. There are truths, at least one.¹

Afterwards, he argues by induction that there are infinitely many truths. The induction step runs as follows. Suppose that there are n truths:

$$[A_1], \dots, [A_n].$$

Now consider the proposition:

[Apart from $[A_1], \dots, [A_n]$, no other proposition is true.]

This proposition differs from each of $[A_1], \dots, [A_n]$. And, as before, it would have to be false if it were true; hence it is false, and consequently its negation (which also differs from each of $[A_1], \dots, [A_n]$) is true. Thus, whenever there are n truths, there are $n + 1$ of them.²

We note that these proofs make essential use of both self-reference and a truth-predicate, a combination that is well-known to be highly combustible.

(D) TEMPORAL DETERMINATIONS

A further complication arises in the case of propositions about actual objects, which in his opinion can change, i.e., have different attributes at different times.³ The basic example is that of a substance and its attributes, a substance being eternal, at least some of its attributes changeable:

¹ **WL**, §31. Bolzano discusses the proof further, offering a direct version, in §530. For a detailed discussion of these matters, see Stefania Centrone, “*Consequenzia Mirabilis*, Antiskeptizismus und Antinomien. Über Bolzanos Beweis, daß es wenigstens eine Wahrheit an sich, daß es der Wahrheiten mehre, ja unendlich viele gebe,” pp. 233–62 in S. Centrone, *Studien zur Bolzano* (St Augustin: Academia Verlag, 2015).

² **PdU**, §13 offers a different proof that there are infinitely many truths provided that there is one along the following lines. Suppose that $[A_0]$ is a truth. Now let $[A_{n+1}] = [[A_n] \text{ has truth}]$, for $n = 0, 1, 2, 3, \dots$. We then have an infinite sequence of truths $[A_0], [A_1], \dots$, each of which differs from the others because it has a different subject-idea. Cf. **WL**, §32, note and **BBGA** 2A.7, pp. 159–60 (§103).

³ For detailed discussion of this question, see M. Textor, “Caius-at-noon, or, Bolzano on Tense and Persistence,” *History of Philosophy Quarterly* **20** (2003) 81–102.

It seems to me to follow from the mere concept of a substance that they can neither come to be nor cease to be. Substances that exist at one time must exist at all times. Only their attributes, the adherences, can come to be or cease to be; whenever there is an alteration, they, the substances themselves, are that which we must represent as the object that undergoes the alteration, and which consequently does not come to be but rather already existed, although it became something else [*ein Anderes*].¹

Grammatical evidence, namely, the feature of tense, suggests that temporal determinations belong with the verb, and thus ultimately with the copula. Bolzano is not convinced, however, holding that here, as elsewhere, grammar is misleading. In support of his position, he notes that we must choose a tense for a verb even when stating a proposition about objects that are not in time. For example, we employ the present tense in saying that two *is* prime, but this is not taken to invite questions such as: “Was it prime last week?”

This shows that we must not conclude that since language connects the concept of having with time determinations, there is an essential connection between them.²

Among the various options available to him, Bolzano settles on that of attaching the temporal determinations to the subject-idea.

[T]hese determinations belong essentially to the subject-idea of a proposition. A proposition of the form: “The object *A*—has at time *t*—the attribute *b*”, if its parts are to be distinctly indicated, must be expressed in the following way: “The object *A* at time *t*—has—(the attribute) *b*.” For it does not happen at time *t* that the attribute *b* is ascribed to the object *A*; but the object *A*, inasmuch as it is thought to exist at time *t* (hence to have this determination) is claimed to have attribute *b*.³

This decision allows him to maintain several theses: first, that the law of non-contradiction does not require a temporal qualification, i.e., he can maintain simply that no object has contradictory attributes, rather than saying that no object has contradictory attributes *at the same time*⁴—this based

¹ *Athanasia*, 2nd edn, p. 79.

² **WL**, §127, no. 5 [II.15].

³ **WL**, §127, no. 5 [II.15].

⁴ **WL**, S 45, no. 1.

on the further claim that when t and t' are different times, $A_{(at\ t)}$ and $A_{(at\ t')}$ are different objects;¹ second, that one and the same *substance* can have incompatible attributes (namely, at different times); and finally that the truth or falsity of propositions does not change.²

(E) RELATIONS

One might wonder how a logic built around the subject–predicate form could accommodate relations. Reasonably well, as it turns out. A relational statement does in Bolzano’s opinion predicate a property of an object, but the object in question is complex, namely, a *whole* or *collection* [*Inbegriff*] with several parts:

It is easy to see that every object will have its own attributes. A whole that has several objects $A, B, C, D, \dots$ as parts is, as such, a special object, which is essentially different from its parts. It is obvious that each whole will have certain attributes which its parts do not have. If I am not mistaken, these attributes are what we call *relations between those parts*. In particular, this holds when we think of the objects $A, B, C, D, \dots$ on one the hand and the attribute x of the whole on the other, as *variable*, i.e., if we think that other objects $A', B', C', D', \dots$ which are of the same *kind* as $A, B, C, D, \dots$ have an attribute that is, although not the same, yet of the same kind as x . Thus, for instance, if the line A is twice as long as B , this is not an attribute that belongs to any one

¹ WL, §45, no. 1 [I.202]: “...an object at another time is actually a different object.” Cf. below, p. 461.

² WL, §125 is entitled: “Every proposition is either true or false, and remains that way always and everywhere.” Arianna Betti takes this section to indicate that Bolzano thought that propositions in themselves, though non-actual, nonetheless have truth or falsity at a time (indeed at all times), and thus admit of temporal determinations. See her “Sempiternal truth. The Bolzano–Twardowski–Leśniewski axis,” pp. 371–99 in J. Jadacki and J. Pańniczek, eds, *The Lvov-Warsaw School—The New Generation*, Poznań Studies in the Philosophy of the Sciences and the Humanities, Vol. 89 (Amsterdam: Rodopi, 2006), p. 374. It seems to us that it is also possible that Bolzano was speaking loosely here, and merely meant to indicate that with propositions, unlike many sentences, a time-indication is not required for truth or falsity. While doing violence to Bolzano’s actual words, this interpretation seems to us to better fit with his remark that temporal determinations are attached only to actual objects (3-D, §2, no. 6 [BBGA 1.18, p. 228]). If this interpretation were correct, propositions would have truth or falsity timelessly.

of the two lines, A and B , taken by itself, but to the whole that consists of both of them. Furthermore, if we replace these lines by others, then the new whole will not always have the same, but only a similar attribute, e.g., that one line is three times as long as the other. Thus, one line's being twice as long as another we call a relation that holds between these lines.¹

Thus we can say that a relation between A and B , e.g., that A is twice as long as B , is a property of a collection containing these two objects as parts—in modern terms, the ordered pair $\langle A, B \rangle$. An idea of this collection will then serve as the subject-idea, and the predicate-idea will represent an attribute of this collection (a relation).

For a more complex example, consider the kind of relations that Bolzano includes under the name *equality*. For him, such a relation will obtain iff given objects $A, B, C, D, \dots$ all stand under certain ideas $i, j, k, \dots$.² A statement that such a relation obtains (an *equation*) is again a statement about a collection containing $i, j, k, \dots$ along with $A, B, C, D, \dots$.³ We can represent this collection in modern terms as an ordered pair of collections:

$$\langle (A, B, C, D, \dots), (i, j, k, \dots) \rangle$$

An idea of this collection, then, is called upon to serve as the subject-idea. The relational statement will then be expressed, in subject–predicate form, along the following lines:

The collection $\langle (A, B, C, D, \dots), (i, j, k, \dots) \rangle$ has the attribute that each of the objects contained in its first part stands under each of the ideas contained in its second part.

For example, suppose that the ideas $i, j, k, \dots$ are the following concepts:

[polygon]	[figure with a right angle]
[three-sided figure]	[figure with two equal angles]

and let $A, B, C, D, \dots$ be objects standing under each of these concepts, i.e., isosceles right triangles. In this case, we say that $A, B, C, D, \dots$ are equal in these respects by making a claim about the corresponding collection.

¹ WL, §80, no. 3 [I.381].

² WL, §91, note 2 [I.430]; cf. GL, Vorkenntnisse, §65 [BBGA 2A.7, p. 117 *et seq.*], and below, p. 442 *et seq.*

³ WL, §135, no. 12 [II.41]. In Bolzano's example, there is only a single idea.

An interesting feature of such examples is that the order of the objects $A, B, C, D, \dots$ and also of the ideas $i, j, k, \dots$ is a matter of indifference. Bolzano speaks in such cases of *mutual* relations or again (in a different sense of ‘equality’) of relations of equality. In such cases, he writes, “the objects $A, B, C, D, \dots$ all make the same contribution to the attribute that belongs to the whole which they form.”¹ A simpler example is the distance *between* two points, as opposed to the direction *from* one *to* the other, which is a relation of inequality.

He continues: “If a relation of equality holds between the objects $A, B, C, D, \dots$, then it must be possible to represent this relation through an idea in which the ideas $A, B, C, D, \dots$ all occur in the same way, i.e., in the same connections.”² That is, if we change the order of terms in an expression designating a relation of equality, it will not change the idea: ‘the distance between A and B ’ thus designates the same idea as ‘the distance between B and A ’.

In modern terms, we could capture this feature by looking upon the collection in question as an ordered pair of *sets*,³ namely:

$$\langle \{A, B, C, D, \dots\}, \{i, j, k, \dots\} \rangle$$

Bolzano notes that his interpretation of such statements is all the more to be preferred in that equality or similarity (for him, a special case of equality⁴), for example, may be predicated of any number of things. In mathematics, for instance, we might want to say that all circles are similar, though not all ellipses are or, in defining a certain kind of integral, that all the intervals in a given partition are of equal length. Bolzano’s interpretation allows one to do this with no difficulty.

The expressions “ A is equal to B ” and “ A is similar to B ” have created the impression that the concepts of equality and similarity are, respectively, the copula in such propositions, while A is the predicate and B the subject. One can already gather that this is wrong, it seems to me, from the fact that similarity and equality may be asserted of more than two, indeed of infinitely many

¹ WL, §80, no. 5 [I.383].

² WL, §80, no. 5 [I.383].

³ Bolzano’s term for a collection in which the order of the parts is considered a matter of indifference is ‘*Menge*’, which we have chosen to translate not as the more familiar ‘set’, but rather as ‘multitude’. Discussed below, p. 430 *et seq.*

⁴ See below, p. 444.

objects. Moreover, there is no reason why one of them should be mentioned in a different way than any of the others in such a proposition. Now if A , say, were to appear as subject, and B as predicate, this would not only be a distinction without any basis, but we would also only be able to declare two things equal or similar in this way.¹

Note, too, that interpreting assertions of equality or similarity as involving sets will give us the means to establish the symmetry of these relations in the binary case, since, e.g., the set $\{a, b\}$ does not differ from $\{b, a\}$.

Other relations will be attributes not of sets but instead of ordered collections. The relation IS OLDER THAN, for example, is not symmetric, and thus 'A is older than B' cannot be interpreted as a statement about the set $\{A, B\}$, but will have instead to be interpreted as a claim about an ordered pair.²

We should not assume, by the way, that the sort of structure we find in ordered n -tuples was the only sort contemplated by Bolzano. The cyclical order exhibited by the points of the compass, for instance, would correspond to a relation among them, but there would be no reason to suppose that there is a distinguished first, second, third, or fourth element. A geometrical object, or structured collection of points, to take another example, will also have a variety of attributes (relations) that are not of this kind, e.g., that of constituting a continuum, or possessing radial symmetry, etc.

Bolzano's approach is not very different from that taken by many contemporary logicians, who distinguish between properties and relations simply by the number of places for terms. The meaning of an n -place predicate constant is fully determined once one has specified an appropriate extension (a set of n -tuples), whether n be 1 or some larger number, and this is also the case for Bolzano, insofar as the predicates in question designate *simple* concepts. The treatment of the two cases is uniform, no sharp line is drawn between one- and many-place predicates on the level of semantics, and it does not seem entirely unreasonable to say that on this understanding a relation is a property of a collection (namely, an ordered n -tuple). If there is a significant difference here, it is the usual one that Bolzano's conception is more liberal than those customarily presented today, permitting relation terms with variable (infinite as well as finite) numbers of places and also unordered as well as ordered collections of terms, where 'order', moreover, is understood in a very broad sense.

¹ WL, §135 note [II.43]. Cf. WL, §91, note 2.

² Cf. WL, §135, no. 13 [II.42].

(F) EXISTENCE AND OBJECTUALITY

The first part of the *Theory of Fundamentals* bears the title: “Of the Existence of Truths in Themselves”.¹ Bolzano later regretted this choice as potentially confusing—understandably, because in that very part he expressly denies that truths in themselves have existence.² The apparent contradiction is resolved by claiming that, in the title, the word is used in an improper sense, later in its proper sense.

In its proper sense, Bolzano thinks, ‘existence’ refers to a property: objects such as planets, rocks, trees, frogs, and dust particles have it, while others, such as propositions and ideas in themselves, and mathematical objects such as geometrical figures, lack it.³ The property in question, which he calls *actuality* or *being* (*Wirklichkeit*, *Seyn*, *Daseyn*), is ascribed to objects; the idea [actuality], therefore, should appear in predicate position, as for example in the claims:

[The Sun exists.] = [The Sun has actuality.]

[Truths in themselves do not (really) exist.] = [Truths in themselves have non-actuality.]

When, by contrast, we claim truly that truths in themselves exist, this is not at all what we mean. When speaking carefully, Bolzano marks the difference by saying not that *there exist* truths but rather that *there are* truths. In such cases, he maintains, we are not ascribing existence or actuality to the truths, but rather ascribing a property to the *idea* of a truth, the property of representing one or more objects, for which he coined the term *objectuality*.⁴

The ordinary usage of ‘exist’ and ‘existence’ thus covers two quite different concepts: actuality and objectuality. Given that there are actual as well as non-actual objects, the inference

[A is actual.]
 —————
 [[A] has objectuality.]

is valid, but the converse fails.

With the concept of objectuality in hand, Bolzano can easily avoid some ancient paradoxes. How, for instance, could one deny that *A* exists, since

¹ WL, §17 ff.

² WL, §25 (b).

³ WL, §142.

⁴ WL, §137.

the truth of the proposition [A does not exist] on the usual interpretation requires that there *be* an A, which has a property (namely, the property of non-existence)? The truth of this proposition seems to presuppose the very thing it denies. On Bolzano's account, nothing could be simpler. To say that there is no such thing as a round square is simply to claim:

The idea [round square] has objectlessness.

And to say it is true presents no problems on the standard definition, for the subject concept does represent something (the *idea* [round square]), which indeed has the property predicated of it (i.e., of not representing anything).

Propositions . . . such as "there is no round square" actually have the following sense: "the idea of a round square lacks objectuality." Hence its subject is again an objectual idea, for only the idea "round square" is objectless; the idea of this idea (which is the subject of the proposition) is an objectual idea. Its object is that first idea.¹

While the foregoing is no doubt old hat to our readers, the following remark of Bolzano's might give them pause:

The true sense of sentences [such as "There is an A"] is . . . simply that there is an object that corresponds to the idea A. Only in the case where it already lies within the idea A that the object corresponding to it is an existing one (e.g., with the concept "God") is the proposition "There is an A" equivalent to "A has existence" (though these two propositions are by no means identical).²

A quick reading of this passage might suggest that Bolzano is here endorsing the validity of the ontological argument. But it is in fact not so. He does maintain that some ideas ascribe actuality to their objects, or at least ascribe properties to their objects from which their actuality follows. The concept of God as defined by Bolzano, for instance, is that of a being (i.e., something which is actual) that has no ground of its actuality. Actuality being part of the concept of God in this way, it would seem to follow from the definition that the proposition [God has actuality] must be true. Recall, however, that this last proposition counts as true only if the subject concept represents an object, which by no means follows from its definition alone. As Bolzano's student Příhonský explained:

¹ WL, §196 [II.329]; cf. §138.

² WL, §137 [II.53].

If someone wanted to draw the inference that God necessarily exists because actuality forms part of the concept of God, he would admittedly commit a fallacy. For it cannot be immediately inferred from the fact that a certain attribute *a* is thought in the subject-concept *M* = Something, which has the attributes *a, b, c, ...* that the proposition: *M* has *a* is true. For the following premise also belongs to that inference: *The idea of something which has the attributes a, b, c, ... has objectuality*.¹

Bolzano's remarks, by the way, indicate that propositions which are analytic in Kant's sense may be false.²

(G) NEGATION

Given the basic subject–predicate form ‘*A* has *b*’, there would appear to be at least three places to insert a negation within any proposition:

(Subject) Non-*A*–has-*b*.

(Copula) *A*–does not have-*b*.

(Predicate) *A*–has-non-*b*.

Ordinary usage suggests that there are propositions of all three of these kinds, e.g.:

Non-members have voting rights.

Socrates hasn't any vices.

Featherstone is unwell.

Bolzano argues, however, that linguistic appearances are again misleading, that the first case only very rarely occurs as displayed above, and the second never. The third case, predicate-negation, is taken pretty much at face value, however. It lies at the centre of his account of negation, and he analyzes all

¹ F. Příhonský, *Neuer Anti-Kant* (Bautzen, 1850), p. 147. Cf. Bolzano's manuscript “Verbess. u. Zus. zur Logik,” **BBGA** 2A.12/2, p. 103: “*Judgments of existence*: The flaw of the ontological proof of the existence of God did not lie in the fact that existence is not an attribute, as Kant had claimed. Rather, the flaw is that, while existence is presupposed of every being [*Wesen*], it must first be proved that the concept of the being is an objectual one.” Also R. George, Editor's Introduction to Bolzano, *Theory of Science* (London: Blackwell, 1972), pp. xxxi–xxxii.

² See below, p. 279.

but a tiny number of sentences containing negation in terms of this sort of negation alone.

The first case is divided into two others, depending upon whether the subject-term of the sentence

Non-A–has–*b*

is taken to designate an *absolutely* or only a *partly* negative idea. Of absolutely negative ideas, he writes:

[These] are contained under the form *non-A*, and simply posit the negation of a certain idea *A*, without requiring that in the place of the negated idea some other idea, not even that of a something, should be thought. . . . I have no doubt that there are at least some of them. For why should we not be able to think by the words “not blue”, “not round”, etc., merely the denial of that which is expressed by the words “blue”, “round”, etc., without having to posit something else in the place of these denied attributes, e.g., the idea of something in general? In any case, I believe that all will admit that at least the idea designated by the word “nothing” is purely negative. For “nothing” clearly just means “not something”, and to claim that with this idea something is tacitly posited amounts to claiming that the idea “nothing” is the self-contradictory idea of something “which is not something”.¹

Absolutely negative ideas are objectless. Thus, even though they do occur as the subject-ideas of some propositions, e.g., [Nothing–has–triangularity], none of the latter is true. For this reason, Bolzano deems such propositions to be of no particular interest.²

Incidentally, most sentences whose subject-term is ‘Nothing’ do not express propositions belonging to this form. For example, the sentence:

Nothing is both round and square

would according to Bolzano express a proposition better stated as follows:³

The idea of something that has roundness and squareness–has–non-objectuality.

The second sub-case of subject-negation involves partly negative ideas, those in which “the negation governs only one or several of their parts.”⁴ A

¹ WL, §89, no. 3 (a) [I.416–17].

² WL, §136, no. 1.

³ Cf. above, p. 260.

⁴ WL, §89, no. 3 (b) [I.417].

common example belongs to the form ‘A, which has non-*b*’, e.g., [Something, which is not blue] or [Triangle, which is non-equilateral]. A sentence such as: ‘Non-equilateral triangles are three-sided’, would, for example, express a proposition better stated as follows:

A triangle, which has non-equilaterality, has trilaterality

where we note that the negation is interpreted as being attached to a predicate-idea.

The second case, or copula-negation, is assimilated to propositional negation, and then reduced to predicate-negation.¹ Thus to say

Socrates isn’t wise

if it is not interpreted to mean

[Socrates–has–non-wisdom]

amounts to *denying* the proposition stating that he *has* wisdom. This denial is in turn interpreted as ascribing a lack of truth to that proposition, as follows:

[[Socrates has wisdom]–has–non-truth.]

Generally speaking, a proposition [*A*–has–*b*] admits of predicate- (or internal) as well as propositional- (or external) negation, namely:

[*A*–has–non-*b*.] [[*A*–has–*b*]–has–non-truth.]²

In much the same way, some modern logicians recognize the forms

$\langle \lambda x. \neg Bx \rangle (a)$ $\neg \langle \lambda x. Bx \rangle (a)$

as negations of $\langle \lambda x. Bx \rangle (a)$ or, more simply, *Ba*.

That it is worth distinguishing the two forms of negation becomes clear from Bolzano’s treatment of propositions with empty (objectless) subject ideas, which, as we saw, he took to be false.³ Thus, for example,

[A free lunch includes drinks]

is false, but so too is its (predicate-)negation

[A free lunch does not include drinks.]

¹ WL, §136, no. 2.

² WL, §189, no. 1(e) [II.269]; cf. §136.

³ WL, §234, no. 3. As are propositions with objectless predicate ideas (no. 4).

Since the subject idea of both is empty, while the proposition-negation

[[A free lunch includes drinks] has non-truth]

is true. As in this example, so too in general, proposition negation produces contradictories, while predicate negation may only produce contraries.¹

Apart from the rare and uninteresting cases where the subject-idea is absolutely negative, then, Bolzano tries to get by with predicate-negation alone. His account of this case, however, gives rise to some ontological questions.² According to Bolzano's definition of truth, a proposition is true iff each object that stands under its subject-idea has an attribute that stands under its predicate-idea. This also holds in the case where the predicate-idea is of the form [non-*b*]. Bolzano's ontology thus seems to include negative attributes, lacks or absences, something he explicitly affirms:

[T]he lack of an attribute is itself an attribute, and the lack of the lack makes the original attribute appear once again.³

In line with this, he thinks that the proposition [*A* has non-*b*] would be more distinctly expressed as follows:

A—has—the attribute Not *b*.⁴

We recall, however, that the predicate-idea in a proposition [*A* has *b*] often represents several attributes.⁵ In light of this, to say that *A* lacks *b*, e.g., that Caius lacks intelligence, is presumably equivalent to denying that he has any of the attributes standing under the idea [intelligence]. Thus the attribute of non-intelligence would belong to Caius just in case he lacked every kind of

¹ Note that this analysis also covers Russell's famous example: "The Present King of France is bald." ("On denoting," *Mind* 14 (1905) 479–93). A further point is perhaps worth mentioning here. While many modern logicians deem singular propositions with empty subjects to be false (or rather, they prefer to eliminate non-referring singular terms altogether), they also claim that general propositions with empty subjects are true. For example, the proposition "The round square is round" counts as false under Russell's analysis, while "All round squares are round" is deemed true. Bolzano's uniform analysis declares all propositions with empty subjects false.

² Ettore Casari has provided the most thorough investigation of these we are aware of in section 18 of his book *Bolzano's Logical System* (Oxford: Oxford University Press, 2016).

³ **WL**, §136, no. 2 [II.47].

⁴ **WL**, §136, no. 2 [II.50].

⁵ Cf. above, p. 251.

intelligence. Indeed, based on the remarks we just quoted, it seems reasonable to suppose that the attribute would *be* precisely that of lacking every kind of intelligence. Furthermore, according to the above remarks, the lack of this lack would not be a further attribute, but rather the simple possession of some kind of intelligence.

Further complexity arises if we assume that predicate ideas of the form [Attribute Not *b*] may also represent several objects. If, for example, we were to say: “All the members of this political party lack integrity,” or, more distinctly, “Member of this political party—has—the attribute non-integrity,” we might well mean that each member of the party has his or her own absence of integrity, where, in each case, the attribute would be that of lacking each and every kind of integrity.

(H) CATEGORICAL STATEMENT FORMS

The concept of objectuality is also at the heart of Bolzano’s treatment of the categorical statement forms of traditional syllogistic. [Some *A* is *B*] is analyzed as [[*A* which has *b*] has objectuality], while [No *A* is *B*] is rendered as [[*A* which has *b*] has objectlessness.]¹

Bolzano notes the (near) equivalence of [No *A* is *B*] and [All *A* are non-*B*], thus opening up the possibility of analyzing [All *A* are *B*] as [[*A* which has non-*b*] has objectlessness.]² He does not do so, however, preferring instead the simpler expression ‘*A* has *b*’. The word ‘all’ in such cases, he claims, is redundant, since the subject idea [*A*] is always taken in its full extension:

I take it that the expression “any man” means no more than what we think by the expression “man” alone, indeed what we must think by the word “man” if we do not want to limit it arbitrarily to one or the other class of men; the only point of the addition “any” is to prevent such a limitation. This is especially necessary if a word is frequently connected with additional ideas which are either expressed by the word “some” or other words, or else tacitly added [...]. It is for this reason that words that are seldom or never used in a narrower sense and words that designate an idea of which we are not used to distinguish subordinate kinds, rarely occur with the words “any” and “every”. Thus, we say that “in any triangle the sum of all internal angles equals two right

¹ WL, §§137, 138.

² Because Bolzano requires for the truth of [All *A* are *B*] that there be at least one *A*, we would also have to add this condition to the above formulation.

angles” since several kinds of triangles are known, and someone might think merely of, say, equilateral triangles if the word “any” were not added. On the other hand, if we speak of right triangles we do not say “in any right ...” but merely “in a right triangle the square over the hypotenuse, etc”. The reason for this is that the addition “any” is here found superfluous, since we do not usually distinguish several kinds of right triangles. Hence there is no reason to fear that someone may think only of a special kind of them (e.g., the isosceles right ones).¹

For example, he thinks that the proposition [All men have mortality] is more distinctly expressed by the sentence ‘Man has mortality’.

This decision accords to a certain extent with a widespread practice of mathematicians, who often do not express initial universal quantification, stating certain theorems with free variables on the understanding that they may be taken to be equivalent to their universal closures. Here are a couple of examples taken at random from mathematical texts:²

- **Theorem.** A non-void subset S of a group G is a subgroup if and only if (i) a and b in S imply ab in S , (ii) a in S implies a^{-1} in S .³
- **Lemma.** If E_1 and E_2 are measurable, so is $E_1 \cup E_2$.⁴
- We assume that the resulting operation $+$ has the familiar properties:⁵

$$0 + a = a \quad a - a = 0 \quad a + b = b + a \quad (a + b) + c = a + (b + c)$$

Though Bolzano thinks that statements of the forms “All A are B ” generally presuppose the existence of A s, he also recognizes that there are cases where existence is not presupposed. There are, he writes,

¹ WL, §57, no. 2.

² Among logicians, Frege followed this practice in his *Begriffsschrift* (§11), as did Russell (*The Principles of Mathematics*, 2nd ed. (New York: Norton, 1937), p. 72.), Carnap, (*The Logical Syntax of Language* (London, 1937), pp. 21–2), and many others.

³ G. Birkhoff and S. MacLane, *A Survey of Modern Algebra* (New York: Macmillan, 1950), p. 143.

⁴ H. L. Royden, *Real Analysis*, 2nd edn (New York: Macmillan, 1968), p. 57.

⁵ J. L. Bell, *A Primer of Infinitesimal Analysis* (Cambridge: Cambridge University Press, 1998), p. 18.

... a very large number of propositions which are considered to be true, but where we do not care whether their subject ideas have an object; in these cases, the existence of such an object depends upon quite accidental extraneous circumstances. For example, "A golden mountain would be bare", "Whoever shows up at this place at such and such an hour must be suspected and should be taken into custody", and others. Closer inspection shows that expressions of this kind, whenever it becomes doubtful whether their subjects really have an object, should only be understood as conditional assertions, roughly in the following way: "If a mountain were made of gold, then it would be bare" and "If somebody were to show up . . . , etc."¹

Existential and Categorical Statement Forms

Sentence form	Bolzano's rendering
There are <i>Ps</i> .	[<i>P</i>] has objectuality.
Nothing is a <i>P</i> .	[<i>P</i>] has non-objectuality.
<i>A</i> exists.	<i>A</i> has actuality.
<i>A</i> does not exist.	<i>A</i> has non-actuality.
All <i>P</i> are <i>Q</i> . (1)	<i>P</i> has <i>q</i> .
All <i>P</i> are <i>Q</i> . (2)	If there were any <i>Ps</i> they would be <i>Qs</i> .
No <i>P</i> is <i>Q</i> .	[<i>P</i> which has <i>q</i>] has non-objectuality.
Some <i>P</i> is <i>Q</i> .	[<i>P</i> which has <i>q</i>] has objectuality.
Some <i>P</i> is not <i>Q</i> .	[<i>P</i> which has non- <i>q</i>] has objectuality.

(I) DISJUNCTIONS AND CONDITIONALS

In the simplest cases, disjunction is dealt with along the following lines: [*A* or *B* or *C* . . .] is analyzed as [The idea of a true proposition among *A*, *B*, *C*, . . . has objectuality] (inclusive disjunction) or [The idea of a true proposition among *A*, *B*, *C*, . . . has objectuality and the idea of a plurality of true propositions among *A*, *B*, *C*, . . . has objectlessness] (exclusive disjunction).² In such

¹ WL, §196 [II.330]; cf. §225, note.

² WL, §181. Note that the variables *A*, *B*, *C*, . . . occurring in these forms are not to be replaced by propositions, but rather by *ideas* whose objects are propositions.

cases, what is expressed is a relation of *material* complementation.¹ Bolzano also recognizes cases of *formal* complementation, where the above relation continues to hold even when the propositions *A*, *B*, *C*, ... are transformed by replacing some of their parts with others. In the proposition [Caius either knows it or he doesn't], for instance, the relation of complementation will persist even if the idea [Caius] is replaced by another such as [Titus], i.e., exactly one of the two disjuncts will be true in any given instance.² In contrast to material complementation, the formal variety is not truth-functional—consider, for example, the propositions [Obama is either a Democrat or a Republican] and [Obama either is or isn't a Democrat], where [Obama] is the only part considered variable.

Conditionals are not truth-functional either according to Bolzano's account. In some cases, he tells us, to say "If *A* then *M*" is to claim that *M* is deducible from *A* with respect to certain variable parts.³ Sometimes, it is fairly obvious which parts are meant to be considered variable. He gives the following example:

If Caius is a man and all men are mortal, then Caius is mortal

where [Caius], [man], and [mortal] are the clear choices for variable parts. With others this is just not the case; for example:

If in all men there is an undeniable striving for continued existence, if even the most virtuous must feel unhappiness at the thought that he shall some day cease to be, then we may rightly expect that God in his infinite goodness will not annihilate us in death.⁴

What is intended with such a statement, he thinks, is a claim of objectuality, namely, that *there are* ideas with respect to which the consequent is deducible from the antecedent.

In §193 of the *Theory of Science*, Bolzano claims a near equivalence between such conditionals and corresponding formal disjunctions, namely, between [If *A* then *B*] and [Either not *A* or *B*], where both of these will involve variation of specified parts. The equivalence is not exact due to his requirement that at least one substitution has to make the antecedent true in order for the conditional to be true. He writes:

¹ WL, §160, no. 2.

² WL, §166, no. 2.

³ WL, §§164, 179; discussed below, pp. 305 f.

⁴ WL, §164 [II.200].

[W]e can certainly deduce from the hypothetical proposition “If A is the case, then B is the case” the disjunctive proposition “Either B or $Neg.A$ is true” and, if we add to this the following propositions “ A is not false after its kind”, then the two last propositions together (they can easily be combined to a single proposition) allow us to deduce the hypothetical proposition in turn.¹

Given that conditionals and formal disjunctions are not truth-functional on Bolzano’s account, one might wonder whether he considers them to express necessary connections. Should conditionals, for example, be interpreted along the lines of strict conditionals in some modal system or other, i.e., as $\Box(P \rightarrow Q)$ for some suitable definition of ‘ $\Box$ ’? Similarly, should we introduce a strict disjunction, $\Box(P \vee Q)$ to try to capture his notion of formal complementation? Bolzano rejects this, making clear his view that the character of necessity is found in some conditionals and disjunctions, but not all:

When the constituents of a hypothetical or disjunctive judgment are pure concepts, or when at least the intuitions occurring in them belong to their variable parts, then both may be combined with a *must* ... without destroying their truth: “If A is the case, then *necessarily* so is B ” and “*Necessarily*, one of the propositions A and B is true.”²

We shall see below, for example, that according to Bolzano’s extremely broad definition of deducibility, the proposition

[Napoleon visited a one-time seat of the Hungarian Royal Court]

is deducible (when the idea [Napoleon] is considered variable) from the proposition

[Napoleon celebrated a famous victory in Pressburg.]

Thus, following what was just said, we can affirm that the conditional

[If Napoleon celebrated a famous victory in Pressburg, then he visited a one-time seat of the Hungarian Royal Court]

¹ WL, §193, no. 1 [II.307]. When Bolzano says that A is not false after its kind, he simply means that at least one substitution transforms A into a true proposition.

² WL, §193, no. 5 [II.312]. See below, 272 ff., for more details on Bolzano’s notion of necessity.

is true (where, again, [Napoleon] is the only variable part). All the same, the corresponding strict conditional

[Necessarily, if Napoleon celebrated a famous victory in Pressburg, then he visited a one-time seat of the Hungarian Royal Court]

is false. By contrast, the conditional

[If 2 is even and prime, then 2 is even]

can be affirmed when [2], [even], and [prime] are all considered variable. In this case, all the constituents of the conditional are pure concepts, so the corresponding strict conditional also holds.

Bolzano's identification of at least some conditionals with statements of deducibility goes against what many of us have had drilled into us in logic class, namely, that we must never lose sight of the distinction between the conditional (an operator belonging to the object-language) and implication (a relation that may obtain between object-language formulae, but which is only expressed in the meta-language). For all that, he is not entirely friendless among the moderns.¹

The kind of conditionals discussed above, however, is not the only one, nor is it even the most common, for there are many cases where the interpretation in terms of deducibility just won't work. Consider one of Bolzano's examples:

[If Caius is dead, then Sempronius is a beggar.]²

It seems clear that no specification of variable parts in the antecedent and consequent will yield a relation of deducibility.³ What we have in mind in stating such a conditional, he thinks, is something like the following:

[T]here are certain relations between Caius and Sempronius such that the following general proposition holds: it holds of any two persons who are related as Caius and Sempronius and one of whom dies (the one in the position of Caius) that the other (the one in the position of Sempronius) must become a beggar.

¹ See, e.g., A. R. Anderson and N. Belnap, *Entailment: the Logic of Relevance and Necessity*, Vol. I (Princeton: Princeton University Press, 1975), pp. 473 ff.

² **WL**, §179 [II.226].

³ Cf. **WL**, §155, no. 21.

As such, it would be a statement of objectuality.¹ In general, making room for properties of the subjects of the antecedent and consequent as well as relations between them, we might interpret ‘If A has b , then C has d ’ as follows:

The idea of attributes p, q , and r such that A has p , C has q , and A, C stand in the relation R , and such that any object that has q , and that stands in the relation R to an object that has p and b , has d , has objectuality.

A modern symbolization of such a claim might look like this (where ‘ $\rightarrow$ ’ designates the material conditional and the universal quantifier has existential import):

$$\exists P \exists Q \exists R ((Pa \wedge Qc \wedge Rac) \wedge \forall x \forall y ((Px \wedge Qy \wedge Rxy \wedge Bx) \rightarrow Dy)).$$

Given that the universal claim is equivalent to a claim of deducibility, namely, that $[C \text{ has } d]$ is deducible with respect to $[A]$ and $[C]$ from the propositions

$[A \text{ has } p]$, $[C \text{ has } q]$, $[\langle A, C \rangle \text{ has } r]$, $[A \text{ has } b]$

we can assimilate this sort of conditional as well to a claim involving deducibility and objectuality.

(J) PURELY CONCEPTUAL AND INTUITIONAL PROPOSITIONS

We have seen above that Bolzano called an idea that is both simple and singular an intuition, and defined a pure concept as an idea that is not an intuition and has no intuitions among its parts. Similarly, he defines a *purely conceptual proposition* as one that contains no intuitions. Propositions containing one or more intuitions are called *intuitional*, *empirical*, or *perceptual*.² The tendentious terminology (*empirical*, *perceptual*) is justified roughly as follows. According to Bolzano, the objects of all human (subjective) intuitions are mental events. To form an intuition is thus to form an idea of an existing particular (namely, the state of mind which is the unique object of the intuition).³ But to do this is to have (a primitive kind of) experience.⁴ Hence a

¹ WL, §179 [II.226].

² WL, §133.

³ WL, §74.

⁴ Cf. WL, §294 [III.115]: “...judgments which contain an intuition, especially those which are held to be true, are usually called *experiences* or *judgments of experience* in a very broad sense of these terms.”

proposition containing an intuition cannot be thought, and still less known by us, without experience.

It follows that any proposition which can be known *a priori* must contain no intuitions—that is, it must be purely conceptual. Note, however, that this would not entail that all purely conceptual propositions (e.g., all propositions of logic and mathematics) can be known *a priori*. For Bolzano maintained only the converse: if a proposition can be known *a priori*, it is purely conceptual.¹

Bolzano's terminology is thus doubly inappropriate, first, because it introduces terms belonging to the theory of knowledge into the discussion of propositions and ideas in themselves and second, because, while it may be the case that intuitional propositions can only be known *a posteriori*, nothing that Bolzano says entails the claim that there are no purely conceptual propositions that are in the same boat (in which case they too might reasonably be called empirical). This being said, there is nothing unsound about his classification, since it is framed purely in terms of the objective features of ideas, namely, content and extension.

(K) NECESSITY, POSSIBILITY, AND CONTINGENCY

Finally, Bolzano has an idiosyncratic way of rendering the modal expressions 'necessarily' and 'possibly'. He notes, to begin with, that these terms are used in a variety of senses, metaphysical, logical, physical, psychological, etc.² Talk of possibility, for example, may have an epistemological import:

We [sometimes] say that something is possible, or can be, if we want to indicate that we do not know any reason for its impossibility or, what comes to the same thing, that no purely conceptual truth which asserts the opposite is known to us. It obviously does not follow from the fact that we do not know such a truth that no such a truth exists; hence we must not confound anything that we call possible in this sense of the word with what is called possible in the proper sense . . .³

In their proper senses, Bolzano thinks that the terms 'necessarily' and 'possibly' only occur in conjunction with the concept of actuality, that is, they

¹ Bolzano was nevertheless confident that many purely conceptual propositions could be known *a priori*. In §133 of the **WL** [II.36], he writes that *most* can. It is not at all clear what he meant by this, or why he said it.

² **WL**, §182 note.

³ **WL**, §182 [II.232–3].

always qualify being. In another sense which corresponds more closely to some of the notions of necessity studied by contemporary logicians, however, Bolzano thinks that necessity, possibility, etc., qualify propositions.

Concerning necessity, to begin with, we may say, for example, that the sum of two odd numbers is necessarily even, or that it is a necessary truth that a triangle has three sides. What we mean by such assertions, Bolzano claims, is simply that these are *purely conceptual truths*, that is, true propositions that contain no intuitions. Depending upon the nature of the concepts involved, one may also speak of metaphysical, physical, psychological, etc., necessity. Similarly, he claims, we say that it is *impossible* for an attribute *b* to belong to an object *A* if the proposition [No *A* has *b*] is a purely conceptual truth. Possibility can then be defined in terms of impossibility: we say that *b* can belong to *A* if [No *A* has *b*] is not a purely conceptual truth. Finally, he interprets assertions of necessity in his strict sense in a similar way: to say that an object *a* exists necessarily is to say that there is a purely conceptual truth [*A* has actuality], where the idea [*A*] represents *a* and *a* alone. Possible and impossible objects are characterized similarly.

We think it fair to say that these definitions are anything but self-explanatory. Still, it seems to us that these concepts are most properly dealt with in the context of Bolzano's metaphysics, so we shall postpone our discussion until then.¹

(L) CONCLUSION

On the whole, there are perhaps more failures than successes in Bolzano's attempted analyses of propositions. The use of the predicate 'true' within his formulations, as later logicians discovered, opens up his system to all manner of problems;² so too, arguably, the failure to distinguish types or language and metalanguage as we see, e.g., in his analysis of conditionals. And even if Bolzano does describe devices with a great deal of expressive power, he nowhere attempts to explain how one might systematically translate the sentences of ordinary language into his preferred idiom.

Bolzano's account has many strong points as well. Whatever the merits of the individual proposals, he did make provision for many of the features of meaning embodied in modern systems of logic: relations, quantification, propositional operators, modal notions, and so on. Though he argued for the

¹ See below, Chapter 8, Section 11.

² Bolzano discusses the Liar paradox in §19 of the **WL**, but seems to think it can be neutralized.

subject–predicate form as one that all propositions belong to, his general conception of form is clearly far removed from that of traditional logic. What is more, it is precisely the logical analyses which permit him to resolve certain longstanding philosophical problems, e.g., solving the riddle of non-being and diagnosing the flaw in the ontological argument. Finally, it is not the least of the merits of Bolzano's account that he himself regarded it as both provisional and incomplete—something almost unheard of among the German logicians of his day, whose claims to infallibility could make a Pope blush.

In the early twentieth-century, it was often said that ordinary language could be an extremely poor guide to logical form and relations. Ryle, famously, spoke of “systematically misleading expressions”, while Wittgenstein claimed that “it was Russell's merit to have shown that the apparent logical form of a proposition need not be its actual logical form.”¹ Volume Two of the *Theory of Science* shows that, here too, later analytic philosophers were tracking in Bolzano's snow.²

7. BOLZANO'S VARIATION LOGIC

(A) INTRODUCTION

In the mid-1930s, prompted in part by Gödel's results on the incompleteness of formal theories of arithmetic and related matters, a number of philosophers, among them Carnap, Tarski, and Quine, formulated almost simultaneously so-called “semantic” definitions of logical consequence and logical truth (sometimes also called analyticity or tautologousness), to go along with the syntactic definitions which had dominated logic since the time of Frege.

¹ *Tractatus*, 4.0031.

² It is surprising, given Bolzano's repeated insistence that ordinary forms of expression do not always accurately reflect the logical components of the propositions they express, to find William Kneale (“Universality and Necessity,” *British Journal for the Philosophy of Science*, **12** (1961) 89–102, p. 93) writing that Bolzano, at least in one context, embraced “the very naive assumption that a proposition must contain distinguishable constituents corresponding to all the distinguishable constituents of a sentence that expresses it . . .” (This is only rarely the case, according to Bolzano.) Similarly, Coffa makes this puzzling remark about Bolzano: “The central idea of logical analysis, the realization that language is an extraordinarily misleading guide to content, was still in the future” (*The Semantic Tradition from Kant to Carnap* [Cambridge: Cambridge University Press, 1992], pp. 39–40).

It was soon noted¹ that Bolzano, a century earlier, had defined remarkably similar concepts in the *Theory of Science*.

Since then, Bolzano has been given due credit for anticipating later definitions of logical consequence, especially Tarski's. At the same time, his work in variation logic is often looked upon as a mere first draft of Tarski's, presenting more or less the same account of logical consequence and related notions, but in a way which is inevitably less rigorous and clear.² Thus Bolzano's variation logic is not widely studied outside the fairly small circle of Bolzano scholars—why, after all, look at the first, stumbling attempts when more refined and polished versions of essentially the same thing are readily available?

Such neglect, though justified in certain respects, is nevertheless misguided. For Bolzano's concept of deducibility differs in many important respects from Tarski's concept of logical consequence, and the differences are not always simple matters of technical improvement. For one thing, Bolzano's concept of consequence—in contrast to the usual historical sequence—is a *generalization* of Tarski's. For this reason and others besides, it is worth taking a closer look at what he has to say.

In order to communicate something of Bolzano's general approach to variation logic, we will begin by sketching his treatment of the concept of analyticity. In the *Critique of Pure Reason* and elsewhere, Kant had defined an analytic judgment as one in which the subject concept contained the predicate concept:

In all judgments in which we think the relation of a subject to the predicate (I here consider affirmative judgments only, because the application to negative judgments is easy afterwards), this relation is possible in two ways. Either the predicate *B* belongs to the subject *A* as something that is (covertly) contained in this concept *A*; or *B*, though connected with concept *A*, lies completely outside it. In the first case, I call the judgment *analytic*, in the second *synthetic*.³

¹ By Heinrich Scholz, *Die Wissenschaftslehre Bolzanos. Eine Jahrhundert-Betrachtung* (Berlin, 1937).

² Stephen Read, for instance, identifies Tarski's rejected substitutional definition with Bolzano's (*Thinking about Logic* (Oxford, Oxford University Press, 1994), p. 41). John Etchemendy does the same, though he acknowledges that this identification is misleading in certain respects (*The Concept of Logical Consequence* (Cambridge: Harvard University Press, 1990), p. 163, note 5).

³ *Critique of Pure Reason*, tr. W. Pluhar (Indianapolis: Hackett, 1996), A6/B10; cf. Kant's *Logik*, ed. Jäsche, §36; *Prolegomena*, §2.

In framing this definition, Kant seems to have had in mind principally universal affirmative categorical judgments, i.e., those of the form: 'All *A* are *B*.' In the Jäsche Logic, he provides the following examples:

1. "Extension (*b*) belongs to every *x* to which the concept of a body (*a+b*) belongs" is an example of an analytic proposition.
2. "Attraction (*c*) belongs to every *x* to which the concept of body (*a+b*) belongs" is an example of a synthetic proposition.¹

It is clear, however, that he intended the concept of an analytic judgment to be somewhat wider, since he claims that his definition is easily extended to deal with negative judgments. Perhaps he also intended it to cover other forms of judgments as well.² Be that as it may, the above definition is the most Kant gives us to work with.

On the face of things, Kant's analytic propositions seem rather trivial. Locke, who had defined something very similar to Kant's analytic judgments for this very reason included them among the "trifling propositions":

Alike trifling it is, to predicate any other part of the Definition of the Term defined, or to affirm any one of the simple Ideas of a complex one, or the Name of the whole complex Idea; as *All Gold is fusible*. For *Fusibility* being one of the simple Ideas that goes into the making up the complex one the sound *Gold* stands for, what can it be but playing with Sounds, to affirm that of the name *Gold*, which is comprehended in its received significance? 'Twould be thought little better than ridiculous, to affirm gravely as a Truth of moment, That *Gold is yellow*; and I see not how it is a jot more material to say *it is fusible*, unless that Quality be left out of the complex Idea of which the sound *Gold* is the mark in ordinary speech. What Instruction can it carry with it, to tell one that which he hath been told already, or he is supposed to know before?³

¹ §36 [Ak 9, p. 111].

² As a remark in the *Prolegomena* suggests (§2; Ak 4, p. 266): "... whatever be their origin or logical form, there is a distinction in judgments, as to their content, according to which they are either merely explicative, adding nothing to the content of the cognition [by which he seems to mean the subject-idea], or ampliative, increasing the given cognition: the former may be called analytic, the latter synthetic." Emphasis added.

³ *Essay*, IV, viii, §5.

For Kant, by contrast, such propositions (or rather judgments) were not always so trifling. For, he claimed, although we have and use a great many concepts, we are not always aware of their constituents. Forming an analytic judgment (at least in certain circumstances) amounts to reporting a successful partial analysis of a concept, and in this sense may be anything but trifling. We see too why Kant could say that the distinction between analytic and synthetic judgments had to do with *content*¹—for, viewed in a certain way, an analytic judgment is a report on the content of a concept (for instance, that the concept of body contains the concept of extension as one of its parts), while a synthetic judgment is not.

In the early work, *Contributions to a Better-grounded Presentation of Mathematics*, Bolzano had simply accepted Kant's definition, at the same time agreeing with Locke's view that analytic judgments so defined were so trivial they hardly seemed worth considering.² He later changed his mind about analyticity, thinking that Kant had hit upon something of real importance but had failed to put his finger on what it was.

In the *Theory of Science* (§148), he criticized Kant's definition, as stated, for being both too wide and too narrow, even if restricted to affirmative categorical judgments. Too wide, because a proposition like the following, which is neither necessary nor knowable *a priori*, would not be counted as analytic by Kant, and yet its subject-idea contains the predicate idea as a part:

[The eldest son of George H. W. Bush, a former president of the USA, is a former president of the USA.]³

But also too narrow, since it would classify as synthetic propositions of the form

A is either B or not- B

which at least seem to be of a piece with the examples of analytic propositions cited by Kant and his followers.⁴

Now there is a ready reply to these and similar counterexamples, namely, that they involve subject- and predicate-ideas of forms not contemplated by

¹ *Prolegomena*, §2 [Ak 4, p. 266].

² **BD**, II, §§17–18 [MW, p. 115].

³ Bolzano's example is: "The father of Alexander, King of Macedon, was King of Macedon." (**WL**, §148 note [II.87]). We will see below that for Bolzano analyticity, at least in the broad sense, implies neither necessity nor apriority.

⁴ In the *New Essays* (Book IV, Chapter ii), Leibniz had already recognized a much wider variety of forms under the heading of "identities".

Kant.¹ But while this may excuse some of the shortcomings of Kant's definition on the basis of the inadequate account of conceptual form he inherited (and accepted) from the logic manuals of his day, they remain deficiencies all the same.

Bolzano saw something more interesting here. Looking at a typical example of an analytic proposition in Kant's sense, for example,

(*) [A man who is married is married]

he recognized that while one could say that its analyticity depended on its content (i.e., the fact that [married] occurs in both subject and predicate), it was far more fruitful to look upon analyticity as a matter of form—here, the propositional form

An *A* which is *B* is *B*.

For him, the interesting thing here was the invariance of truth-value under variation of some parts of a proposition: not only is (*) true, it also remains true whenever we substitute appropriate ideas for [man] and [married]. Reversing Kant's point of view, Bolzano claims that the analyticity comes about because the truth of the proposition is in a certain sense *independent* of some of its content:

I believe that th[e] importance [of analytic propositions] lies in the fact that their truth or falsity does not depend upon the individual ideas of which they are composed, but that it remains the same irrespective of the changes to which some of their ideas are subjected...²

Later logicians would say that the ideas [man] and [married] occur *vacuously* in the above proposition.³

At this point we must touch upon a point of detail. Clearly, no matter what propositional form we are talking about, not all substitutions will result in true propositions. In the proposition

[The man Caius is mortal]

¹ Kant, as Frege remarked (*Foundations of Arithmetic*, §88), seems to have looked upon concepts as simple sums of characteristics.

² **WL**, §148 [II.88].

³ Or, rather, that these *expressions* occur vacuously in the *sentence*. See, for example, W. V. O. Quine, "Truth by convention," in *The Ways of Paradox and Other Essays* (Cambridge: Harvard University Press, 1976), p. 80.

for instance, substitution of the idea [Titus] for [Caius] will result in another true proposition, but the substitution of, say, [Vladivostok] will not.

In Bolzano's opinion, this problem arises even in the case of propositions such as

[A married man is married]

since the substitution of [round] for each of the two occurrences of [married] and [square] for [man] results in the proposition:

[A round square is round]

which he counts as false because its subject-idea represents no object.¹

Recall that Bolzano uses the term 'objectual' for ideas that have objects and the term 'objectless' for those that do not. A *proposition* is said to be objectual if its subject-idea is. He then proposes the following modification to the above remarks. What is special about propositions which are analytic in Kant's sense, i.e., those of the form

An A , which is B , is B

is that every *objectual* instance of the form is true.

With this proviso in mind, we can say that one of the most interesting features of Kant-analytic propositions is that their truth-value remains constant under an entire class of transformations. Once we see things in this light, however, there is no reason to single out Kant's case (All $A + B$ are B) for special attention. For invariance of truth-value under a class of transformations is clearly a far more general phenomenon. Propositions such as

[If $e < 3$ and $3 < \pi$, then $e < \pi$]

for example, also have this property, since uniform variation of all occurrences of $[e]$, $[3]$, and $[\pi]$ does not change the truth value, provided that the resulting propositions are objectual. So too the proposition

$[2 < 3 \text{ and } 3 < 2]$

is not only false, but remains so no matter what ideas we may substitute for $[2]$ and $[3]$.

¹ Cf. WL, §305, no. 1. Examples such as [A round square is round], incidentally, show in Bolzano's opinion that Kant's claim that all Kant-analytic propositions can be known to be true *a priori* is false. As we have seen, this observation has important implications for the ontological argument, among others. Cf. p. 260, above.

Considerations such as these prompted Bolzano to give a new definition of analytic propositions. A proposition is analytic, namely, if it contains ideas which can be arbitrarily varied without changing the truth-value (subject to the objectuality constraint).¹

It should be noted that Bolzano's extremely general definition covers many propositions that are neither necessary nor knowable *a priori*. The following proposition, for instance, would qualify as analytic in Bolzano's general sense, because it contains an idea, namely [Napoleon], which can be arbitrarily varied without change of truth-value (provided the resulting propositions are objectual):

[Napoleon, who was the Emperor of the French in 1812, was victorious at the Battle of Marengo.]

This is so because every objectual instance of the following propositional form is true:

x, who was Emperor of the French in 1812, was victorious at the Battle of Marengo.

Commentators have on occasion pointed to examples such as these to suggest that Bolzano's definition must somehow be flawed, and that he missed the true essence of analyticity. We think, rather, that he understood very well what others, especially Kant and his followers, wanted to make of analyticity, and, after careful consideration, judged that their views were not tenable. It was, Bolzano believed, a genuine and interesting logical phenomenon that had prompted Kant to introduce the notion of analyticity, only he thought that the phenomenon was not at bottom epistemological, and did not even have a tidy epistemological status. Instead of being a failed attempt to appreciate a Kantian insight, Bolzano's definition should rather be seen as an invitation to look at the question from an entirely different, and far more general, perspective. We will see in the sequel that similar remarks apply to Bolzano's other logical notions, in particular to his concept of consequence, or deducibility.

Bolzano comes somewhat closer, perhaps, to Kant's original sense of 'analytic' when he defines a narrower concept, which he calls *logical analyticity*. A proposition is analytic in this narrower sense if there are ideas in it which can be arbitrarily varied without change of truth value (provided objectuality is preserved), and all the ideas which remain invariable are *logical concepts*.² Kant's analytic propositions are of this sort, for in the form

¹ WL, §148, no. 1 [II.83].

² WL, §148, no. 3 [II.84].

b belongs to every x to which $(a + b)$ belongs

the only invariable parts are logical concepts. By contrast, in the previous example, the relevant form is

x , who was Emperor of the French in 1812, was victorious at the Battle of Marengo.

And here a great many non-logical ideas remain invariable.

It should be borne in mind that here as elsewhere in his logical theories, Bolzano frames his basic definitions for propositions and ideas in themselves, and hence without any reference to the mind, its faculties, operations, etc. Thus there is no epistemological component in these definitions. On the other hand, it is clear that Bolzano thought that a great many epistemological properties could be explained on the basis of the *content* of what is known, i.e., on the propositions and ideas in themselves which are the matter of given thoughts. In particular, the distinction between analyticity in the broad and narrow senses might well be expected to explain why in some cases, but not in others, analyticity can be recognized *a priori*. Of this, more later.

What we see in Bolzano's treatment of analyticity we shall see across the board in his development of variation logic. Beginning with Kant's definition and his examples, Bolzano first identifies a key feature of Kant-analyticity, and then immediately generalizes. Where the original concept was tied to a particular form of propositions, Bolzano formulates his so that it applies to all sorts of propositional forms. Later, we shall see that he also generalizes in another direction, conceiving of analyticity as a limit case of absolute probability. The breadth of logical vision is breathtaking, especially given that nothing in the previous literature is even remotely like Bolzano's theories.

(B) VARIATION, SUBSTITUTION, AND FORM

Before proceeding, we must attend to a number of details. We have seen that Bolzano asks us to consider certain parts of propositions *variable*, and to *substitute* other ideas for these variable parts. What, exactly, does he mean? Concerning variation, to begin with, it might easily seem that there is a serious problem with what he says. For he constructs his account of analyticity for propositions in themselves. Since these are unchanging abstract objects, it makes no sense to speak of them varying in any way. Anticipating objections on this point, Bolzano makes it clear from the start that any such talk is purely figurative:

I hope that no one will object to the expression “variable” and its cognates and that it will not be assumed that what I said contradicts the assertion that ideas are not something that exists. From the latter it indeed follows that they cannot undergo genuine variation. If I say that in a given idea, e.g., “a wise man”, a constituent, e.g., “man” is to be envisaged as variable and may be replaced by any other idea, this merely means that we are to survey all ideas that have the same constituents and order of constituents as “a wise man”, except that they have other ideas in the place of “man”. Thus we are not actually concerned with a *variation* in the proper sense of the word.¹

To speak of variation in such contexts is thus a mere manner of speaking, a shorthand for considering a certain collection of closely related ideas, propositions, or arguments, etc. Similarly, to say that one idea is *substituted* for another is just a convenient way of saying that while we formerly considered one proposition (e.g., [Socrates has wisdom]), we now consider another, related one (e.g., [Aristotle has wisdom]).

Together, a proposition (or idea, argument, etc.) and a class of substitutions determine a class of propositions (ideas, arguments, etc.) which, as we have seen, Bolzano also calls a *form*.² Thus from the proposition [Not all politicians are crooks] and the specification of [politician] and [crook] as variable, we determine a class of propositions including [Not all primes are odd], [Not all dogs are poodles], etc. A linguistic expression containing signs for variables (another sense of ‘form’ used by Bolzano) can, in favourable circumstances, do the same work—here, e.g., the expression ‘Not all A are B.’

Let us now turn to the kinds of substitutions envisaged by Bolzano. To begin with, a stray remark in §223 of the *Theory of Science* indicates that Bolzano sometimes wants to consider entire propositions variable in his variation logic.³ The simplest thing to do in light of this would be to amend his definitions so that propositions as well as ideas might be considered variable. Even with the original definitions, however, there is a way to do propositional logic in his system. For recall that he allows the use of a truth predicate, and that in a proposition [Proposition [A] has truth], or, more simply [[A] has truth], the subject-idea is an *idea* of the proposition [A]. [[A] has truth] can

¹ WL, §69, note 2 [I.314]; cf. GL, Vorkenntnisse, §25 [BBGA 2A.7, p. 111].

² Cf. above, p. 246.

³ II.393.

then serve as a proxy for $[A]$, and we may vary the *idea* of the proposition $[A]$ occurring within the former proposition when we wish to obtain a result equal in all the relevant respects to varying the proposition itself.¹ Recall, too, that in Bolzano's analyses of propositional negation and disjunction, i.e.,

$[\text{Not-}A] = [[A] \text{ has non-truth}]$

$[A \text{ or } B] = [\text{The idea of a true proposition among } [A], [B] \text{ has objectuality}]$

we have to do with *ideas* of propositions $[A]$ and $[B]$ rather than the propositions themselves. By varying these ideas, then, we can do propositional logic in his system, even if we only allow ideas to vary.

Next, we need to consider how to deal with cases where one and the same idea occurs repeatedly in a given logical situation. Tarski, in giving a preliminary definition of logical consequence which he later abandons, describes one model of substitution:

If, in the sentences of the class K and in the sentence X , the constants—apart from purely logical constants—are replaced by any other constants (like signs everywhere replaced by like signs), and if we denote the class of sentences thus obtained from K by ' K' ', and the sentence obtained from X by ' X' ', then the sentence X' must be true provided only that all the sentences of the class K' are true.²

On this account, we are to divide the vocabulary of a given language into two categories, logical and non-logical. We then vary all the non-logical elements, leaving all the logical elements fixed, and taking care to replace signs of the same type with signs of the same type. Thus there are three central features in Tarski's notion of substitution: (1) *All and only* non-logical elements are varied; (2) due to (1), the substitutions are *exhaustive*, in that if one occurrence of a given type of sign is variable, all of them are; and (3) the substitutions are *uniform*, i.e., if more than one occurrence of a given type is variable, all must be replaced by occurrences of one and the same type.

¹ Bolzano himself says (WL, §446) that $[A]$ is equivalent to $[[A] \text{ has truth}]$ —but we can't appeal to that claim here, since its truth presupposes the possibility of allowing an entire proposition to vary, which is precisely what we are trying to avoid. This remark does indicate, however, that Bolzano saw no problem with varying entire propositions in practice.

² A. Tarski, *Logic Semantics, Metamathematics*, 2nd edn (Indianapolis: Hackett, 1983), p. 415.

We shall argue that neither of the first two features belongs necessarily to Bolzano's notion of substitution, and that there is no compelling reason to believe that the third does. With respect to the first, nothing that Bolzano says seems to preclude the variation of some logical concepts.¹ Nor, clearly, does he think it is necessary to vary *all* non-logical concepts. A proposition like [The man Caius is mortal], for instance, he declares to be analytic in view of the universal validity of the propositional form 'The man x is mortal', in which the non-logical ideas [man] and [mortal] remain invariant. Let us turn now to the second and third features. Recall that on Tarski's account substitution is to be both *exhaustive* and *uniform*, i.e.:

- If one occurrence of a given type is varied, then all are; and
- Occurrences of the same type are always to be replaced by occurrences of the same type ("like signs everywhere replaced by like signs").

Thus, for example, in a sentence such as 'Cato killed Cato', the application of Tarski's method could produce variants such as 'Caesar killed Caesar', 'Cato loved Cato', or 'Caesar loved Caesar', but not 'Cato killed Caesar'. Similar remarks apply, *mutatis mutandis*, to the formation of sentential functions from given sentences.

Now Tarski's method of substitution is not the only one to be met with in the classical literature. In his *Begriffsschrift*, Frege describes a more flexible method for generating sentential functions:

If, in an expression (whose content need not be assertible), a simple or a complex symbol occurs in one or more places and we imagine it as replaceable by another (but the same one each time) at all or some of these places, then we call the part of the

¹ Künne claims ("Analyticity and logical truth: from Bolzano to Quine," pp. 184–249 in M. Textor, ed., *The Austrian Contribution to Analytic Philosophy* [London: Routledge, 2006], p. 237, note 63) that [has] is not allowed to vary, but the remarks he refers to (from **ML**, §8, no. 1 [**BBGA** 2A7, p. 62; **MM-EX**, p. 53]) seem to apply only to the main copula in a given proposition, not to secondary occurrences of [has] in the subject or the predicate. For instance, if a proposition were of the form 'A, which has b , has c ', one might consider the entire subject-concept variable, in which case the first occurrence of [has] would also vary, at least as a part of the subject-idea. It is easy, by the way, to see why Bolzano would not want to consider the main copula of a proposition variable—for given his assumptions concerning propositional form, if it were replaced by anything else, the result would no longer be a proposition. Cf. **WL**, §127.

expression that shows itself invariant a function and the replaceable part the argument.¹

Thus on Frege's dispensation, substitution is uniform, but not necessarily exhaustive, in that it allows for the consideration of some, but not all occurrences of a given type as variable. Thus from 'Cato killed Cato', Frege would also allow us to obtain the functions ' ξ killed Cato' and 'Cato killed ξ ', and hence, by substitution, to obtain the sentences 'Cato killed Caesar' and 'Caesar killed Cato'.

Bolzano's remarks on variation are not always as precise as those of Tarski and Frege quoted above, so that, taken in isolation, they might be interpreted to favour either of these two models or perhaps some other. Commentators on Bolzano's variation logic often reflect this uncertainty, in that one sometimes cannot tell which interpretation they favour. We think it fair to say, however, that when a stand has been taken on this point, it has more often than not been in favour of the Tarskian approach. One of the earliest English-language writers on the subject, Yehoshua Bar-Hillel, clearly interpreted Bolzano in this way, as does Jan Berg, the author of the classic *Bolzano's Logic* and editor of the critical edition of the *Wissenschaftslehre* in the Bolzano-Gesamtausgabe.²

Most explicit, perhaps, is Mark Siebel, who writes:

A fundamental assumption [of Bolzano's variation logic] is that a variable idea which occurs several times in one or more propositions is to be replaced everywhere that it occurs by one and the same new idea.³

Siebel argues for the uniformity requirement based upon the claim that without it Bolzano would be forced to retract a number of claims concerning particular cases of analyticity, deducibility, etc. For example, if we were allowed to substitute different ideas for the two occurrences of [dog] in [A dirty dog is a dog] we could easily obtain false variants (e.g., [A dirty cat is an elephant]), so that this proposition would not qualify as analytic, while presumably Bolzano would want to say that it should.

¹ *Begriffsschrift*, §9 tr. Bynum.

² See, Y. Bar-Hillel, "Bolzano's Propositional Logic," *Archiv für mathematische Logik und Grundlagenforschung* 1/3 (1952) 305–38, p. 310; Jan Berg, editor's introduction to *BBGA* I.12/1, p. 16.

³ M. Siebel, *Der Begriff von Ableitbarkeit bei Bolzano*. Beiträge zur Bolzano-Forschung 7 (St Augustin: Academia, 1996), p. 64.

The claim that *all* occurrences of a given idea must be replaced is argued for on quite different lines. In effect, Siebel maintains that to speak of multiple occurrences of an idea-type is absurd:

Morscher points out that speaking of “places where an idea occurs” can only be understood metaphorically. An idea in itself, since it does not belong among the Strawsonian “particulars”, has no place in a “system of temporal and spatial relations.” From this it also follows that an idea cannot actually occur several times in one or more propositions. Speaking of linguistic signs, one can meaningfully say that the same sign occurs twice in different places, as the sequence of signs “false” does in the sentence: “Your thoughts are false, as false as your teeth.” But ideas in themselves cannot be individuated by indicating places or times. It is accordingly absurd to speak literally of several occurrences of an idea-type, which differ only with respect to their place within a proposition. There exists exactly *one* of each idea in itself.¹

But if there cannot be several occurrences of a given type of idea in itself, Siebel argues, it makes no sense to speak of substitutions affecting some, but not all of them. To replace one is to replace them all, and this guarantees both exhaustiveness and uniformity. It is similar to replacing a letter in a crossword puzzle, in a case where it occurs in a down as well as an across entry.

There is some textual support for some of Siebel’s claims. In §91 of the *Theory of Science*, Bolzano asks whether there can be two exactly equal ideas in themselves. The answer he gives there is no:

[A]re there two completely equal ideas? In my opinion, this must be denied, if by ideas we mean ideas in themselves and not subjective (thought) ideas. For one can certainly claim that there may be several, indeed infinitely many subjective ideas that are mutually equal. For one calls such ideas equal if they have the same idea in itself as their matter. Of course, they may still differ in many other respects, for instance their clarity, duration, vividness, or only in that they are found in the consciousness of different thinking beings. But it would be absurd, it seems to me, to speak of two exactly equal objective ideas. For in this case we

¹ M. Siebel, *Der Begriff von Ableitbarkeit bei Bolzano*, pp. 64–5.

consider nothing but the idea in itself, and hence we cannot say that they are equal except when all their recognizable properties (components, mode of composition, etc.) are identical. But if this is the case, then we cannot distinguish them, hence cannot claim that they are several in number.¹

On the other hand, in §274 of the *Theory of Science*, Bolzano writes:

It is undeniable that one and the same objective idea can appear repeatedly in many propositions. Thus in the proposition “Every equilateral triangle is also equiangular” the concept of equality as well as that of angle occurs twice.²

Thus he claims on the one hand that there are no two completely equal objective ideas, while in §274 he claims that the same objective idea occurs several times in a given proposition. How can this be if there is only one? Wolfgang Künne has called this the *repetition problem*.³ Bolzano, as Künne points out, noticed this problem himself when reviewing his own work. In 1835, he wrote:

“There are no two completely equal ideas,” I said elsewhere [*sc.* **WL**, §91]. On the contrary, there are infinitely many completely equal ideas in themselves. For each can be thought in infinitely many combinations. Thus, for example, the concept of equality certainly occurs twice in the concept of a rhombus, a figure with equal sides and unequal angles. Do I not myself speak of combining the concept of negation *with itself*? How would this be possible if one didn’t have an infinite number of negations? What is required for the multiple being of one idea? One and the same idea can only exist once, by the principle of the identity of indiscernibles. There cannot be several equal ideas, etc., because there would be nothing to distinguish them.⁴

¹ **WL**, §91.

² **WL**, §274 [III.16].

³ W. Künne, “Propositions in Bolzano and Frege,” *Grazer phil. St.* **53** (1997) 203–40, pp. 223 ff.; see also W. Künne, “Constituents of concepts: Bolzano vs. Frege,” pp. 267–85 in A. Newen, U. Nortmann, and R. Stuhlmann-Laeisz, eds, *Building on Frege* (Stanford: CSLI Publications, 2001), pp. 278–80. Cf. P. Simons, “Bolzano, Tarski, and the limits of logic,” *Philosophia Naturalis* **24** (1987) 378–405, p. 403.

⁴ “Verbesserungen und Zusätze zur Logik” (manuscript), **BBGA** 2A.12/2, pp. 148–9.

Drawing on a suggestion Bolzano made concerning multiple occurrences of *subjective* ideas, Künne suggests that a single idea in itself be thought of as present only once, but as playing several *roles* in a given proposition.¹

It seems to us that, on Bolzano's principles, a different response is also possible. In §79 of the *Theory of Science*, he examines Kant's claim that space is a pure intuition. One of the arguments he addresses, due to Johann Schultz, runs as follows:

If the idea of space were based upon a concept, then it would be impossible for the geometer to think two congruent areas, for his concept of one of them would be exactly like his concept of the other.

Bolzano replies:

[Schultz] presupposes that it is impossible to form a concept of two or more exactly equal objects, and that such objects must be represented through intuitions. I take this to be an error which refutes itself as soon as it is asserted. For when it is said that it is impossible to represent through concepts several exactly equal things, then several things are talked about, hence represented, and their ideas are certainly not intuitions but mere concepts. Hence the very act of asserting the preceding proposition contains a proof that several exactly equal things can be represented through mere concepts. It may be asked by what characteristics the several exactly equal things can be distinguished in our ideas; I reply that they are distinguished through the differences in their *relations* to each other.²

Thus in the case of abstract objects like those studied in geometry, Bolzano countenances the existence of objects which cannot be distinguished by their inner attributes (this is why they are called *exactly* equal), but which can be distinguished by the relations they enter into with other objects.

It should also be recalled here that Leibniz never meant the principle of the identity of indiscernibles to apply to abstract objects, but only to objects which really exist.³ And—despite what he says in the notebook passage quoted above—Bolzano seems in his more considered moments to agree with

¹ "Propositions in Bolzano and Frege," p. 226; cf. **WL** §274.

² **WL**, §79 [I.376–7].

³ See, e.g., "First truths," in L. Loemker, tr., *G. W. Leibniz: Philosophical Papers and Letters* (Dordrecht: Kluwer, 1989), p. 268: "There cannot be two individual

this view. In §74 of the *Theory of Science*, he accepts the application of the principle to actual objects:

It is indeed false that there are even as many as two actual things that are completely equal to one another in all their (inner) attributes.¹

While in §79, he effectively admits that it does not apply to abstract objects:

It will be recalled that, just like LEIBNIZ, I find a proof that space is not actual [...] in the fact that we cannot determine a single point in space through mere concepts, since they are all completely equal to each other.²

Now if we consider an idea in itself on its own, we do not consider any relations it may bear to other objects. Hence the claim that there can be no two completely equal ideas in themselves is appropriate in this respect. On the other hand, if we consider a proposition like

[An equilateral triangle is also equiangular]

we can distinguish the two occurrences of [equal] by their relations—for instance, by noting that one occurs in the subject-idea, the other in the predicate-idea. Hence it would not seem absurd on Bolzano's principles to speak of two or more occurrences of a given idea in a proposition in itself or an argument, etc.

To the extent that one finds this solution to the repetition problem acceptable, Siebel's argument in favour of regarding Bolzano's notion of substitution as exhaustive would fail. In any case, however, Siebel's argument would at most have shown that it would have been *inconsistent* for Bolzano speak of non-exhaustive substitution. It would by no means show that Bolzano did not

things in nature which differ only numerically. For surely it must be possible to give a reason why they differ, and this must be sought in some differences within themselves. [...] Never are two eggs, two leaves, or two blades of grass in a garden to be found exactly similar to each other. So perfect similarity occurs only in incomplete and abstract concepts, where matters are conceived, not in their totality but according to a certain single viewpoint, as when we consider only figures and neglect the figured matter. So geometry is right in studying similar triangles, even though two perfectly similar material triangles are never found."

¹ WL, §74 [I.333].

² WL, §79 [I.375].

allow such substitutions.¹ In fact, there is good evidence for the claim that he did envisage non-exhaustive substitutions. In §447 of the *Theory of Science*, he writes:

In the broader sense defined in §148, a proposition is called analytic if even one single constituent occurs in it which can be exchanged for any other without affecting the truth or falsity of the proposition, provided only that one chooses an idea which does not result in the proposition becoming objectless. In this sense I must also call propositions such as the following analytic: [...] If $\frac{a^2}{2} = b$, then $a = \pm\sqrt{2b}$. For in [this] proposition there is an idea ..., namely, 2, which can be exchanged for any other without affecting the truth of the proposition.²

Bolzano claims, then, that the proposition

$$(1) \text{ [If } \frac{a^2}{2} = b, \text{ then } a = \pm\sqrt{2b}]$$

is analytic, since it is universally valid with respect to the idea [2]. Shortly afterwards, he points to the following proposition as a *synthetic* one from which the above proposition follows:

$$(2) \text{ [If } \frac{a^2}{c} = b, \text{ then } a = \pm\sqrt{cb}.]$$

And here we note that (2) still contains an occurrence of the idea [2].

Bolzano, unfortunately, does not explain why he thinks the latter proposition is synthetic. It seems to us that there are three possible explanations. Before proceeding to these, however, let us first remark that the letters *a*, *b*, and *c* which occur in the above statements seem to us to function already as variables, which are to be understood as if they were bound by initial universal quantifiers, i.e., (1) should be understood to mean the same as its universal closure:³

$$(1') \text{ [For all } a \text{ and all } b, \text{ if } \frac{a^2}{2} = b, \text{ then } a = \pm\sqrt{2b}]$$

and similarly for (2). If this interpretation is correct, then *a*, *b*, and *c* are not candidates for variation. Let us also note that the only plausible choice for

¹ Cf. **WL**, §375 [III.475–6], where Bolzano speaks of replacing an idea *everywhere* that it occurs, which seems to indicate that non-exhaustive substitutions are at least conceivable for him.

² **WL**, §447 [IV.115].

³ Cf. 265 above.

a variable idea in (2) is the idea [2]. For it is fairly obvious that no other selection of variable parts among the ideas of exponentiation, division, multiplication, equality, etc., would produce a universally valid form either. Thus if (2) is not universally valid with respect to the idea [2], it won't be analytic.

With these preliminaries in mind, the first possible explanation of Bolzano's claim that (2) is synthetic, as we see things, is that he might not have thought of the radical as containing another occurrence of the idea [2] as a tacit index (we generally do not write ' $\sqrt{}$ ' for a square root, though we do write ' $\sqrt[3]{}$ ' for a cube root, ' $\sqrt[4]{}$ ' for a fourth root, and so on). If so, we can see why he would call the above proposition synthetic, since for $n \neq 2$, there will be values of a, b, c which make the following proposition false:

$$(3) \text{ [If } \frac{a^n}{c} = b, \text{ then } a = \pm \sqrt[n]{cb}.]$$

This explanation would clearly presuppose a certain amount of carelessness on Bolzano's part.

It is also possible that he thought that the notation ' $\pm \sqrt[n]{}$ ' indicated that there are always two real roots, which differ only in sign (the only apparent exception occurring when there is a double root at zero), something which occurs only for $n = 2$. If so, then

$$(4) \text{ [If } \frac{a^n}{c} = b, \text{ then } a = \pm \sqrt[n]{cb}]$$

would be false for other values of n , and hence (2) would not be universally valid with respect to the idea [2].

A third explanation would be that Bolzano was thinking of complex as well as real values. One reason to think this is that a couple of pages later, in the very same section,¹ he cites De Moivre's theorem, which is often used for calculating complex roots. We would write this theorem as follows:

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx.$$

If Bolzano was indeed thinking of complex numbers, then, even if we made the index in the radical explicit, by writing either

$$(5) \text{ If } \frac{a^2}{c} = b, \text{ then } a = \pm \sqrt[2]{cb}$$

or

$$(5') \text{ If } \frac{a^2}{c} = b, \text{ then } a = \pm (cb)^{\frac{1}{2}}$$

¹ WL, §447, no. 4 [IV.118].

it would still be reasonable to deny that the resulting propositions are analytic with respect to the idea [2]; that is, to deny that the formulas

$$(6) \text{ If } \frac{a^n}{c} = b, \text{ then } a = \pm \sqrt[n]{cb}$$

and

$$(6') \text{ If } \frac{a^n}{c} = b, \text{ then } a = \pm (cb)^{\frac{1}{n}}$$

are valid for all n .

For, supposing that the notation ' $\pm \sqrt[n]{cb}$ ' indicates that there are at most two roots, and in cases where there are two, the roots differ only in sign, propositions (5) and (5') express something which is peculiar to the case $n = 2$. While it is true that for $n = 2$, the roots come in such pairs (e.g., $\sqrt{4} = \pm 2$, $\sqrt{-9} = \pm 3i$), this does not hold in general. For example, let $n = 3$ and $b = c = 1$ in proposition (6). We obtain:

$$\text{If } a^3 = 1, \text{ then } a = \pm \sqrt[3]{1}.$$

But there are three complex cube roots of unity, namely $1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$, and no two of them differ only in sign.

Whatever the correct explanation may be, there can be no denying that neither the exponent nor the index of the radical is envisaged as variable in this case. We therefore have a clear case of Frege-style variation, where some but not all occurrences of a given type of idea are varied.

The advisability of allowing such non-exhaustive variation is obvious from many mathematical examples. Consider, for example, the universally valid formulas

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$(a + b)^2 = a^2 + 2ab + b^2.$$

If we were compelled to use exhaustive substitution, we would be stuck with saying that while the following propositions are analytic (namely, universally valid, the former with respect to the idea [3], the latter with respect to [3] and [4], with all occurrences uniformly varied):

$$[\sin^2(3) + \cos^2(3) = 1]$$

$$[(3 + 4)^2 = 3^2 + 2 \cdot (3 \cdot 4) + 4^2].$$

These next ones are not:

$$[\sin^2(2) + \cos^2(2) = 1]$$

$$[(2 + 2)^2 = 2^2 + 2 \cdot (2 \cdot 2) + 2^2]$$

because neither of the equations

$$\begin{aligned}\sin^x(x) + \cos^x(x) &= 1 \\ (x+x)^x &= x^x + x \cdot (x \cdot x) + x^x\end{aligned}$$

holds for all x .

Furthermore, it should be recalled that logicians generally make use of linguistic or symbolic *expressions* in order to specify classes of propositions, arguments, etc. Now these expressions can certainly contain multiple occurrences of signs which designate the same idea in itself. So even if, as Künne suggests, there is only a single idea in itself playing several roles in a proposition or argument, we can still consider some, but not all of these roles variable by making use of an appropriate expression. Thus from the expression

A short man met a tall man

we can generate the following *form* (i.e., *form* in the sense of linguistic expression with letters designating variables)

A short X met a tall man

which determines the collection of propositions (i.e., *form* in the sense of species) containing, among others

[A short dog met a tall man.]

[A short cat met a tall man.]

etc.

but also

[A short man met a tall man.]

If it is objected that the above collection cannot be generated from the original proposition by uniform, exhaustive substitution, the response is easy: it doesn't have to be. Our task is to consider a given collection of propositions, not to generate it in the way described.

The above collection, by the way, could also be characterized as the set of propositions which can be generated by (trivially) uniform and exhaustive substitution for the idea [cat] in the proposition [A short cat met a tall man.] Given this, we can certainly say that [A short man met a tall man] belongs to the form 'A short X met a tall man.'

Finally, concerning the *uniformity* requirement, there does not seem to be enough textual evidence to decide the question one way or another. It is obvious that Bolzano on many occasions tacitly supposes that the substitutions will be uniform. On the other hand, it might be useful in some situations to consider non-uniform substitutions, at least when passing from individual propositions to forms. We might, for instance, wish to consider an equation such as $[2=2]$ under the general form of equations ' $a=b$ ', and in this case the transition from instance to form would not be effected by means of a single uniform substitution (though it could of course be obtained in two steps).¹ What is important here, it seems to us, is that, even if we were to allow non-uniform substitutions in some cases, we would reserve the right of specifying in others that certain substitutions are to be uniform.²

* * *

Other restrictions on substitution have to do with what we might, following Husserl, call *semantic category*.³ The thought here is that meanings and partial meanings naturally fall into different kinds, and that in substitution an idea of a given kind should always be replaced by another idea of the same kind. If the boundaries between these kinds are not respected, the result is often something that does not seem to be a meaning at all, just as the violation of grammatical categories in substituting expressions for each other in sentences can result in nonsense.

Bolzano touches on this topic in a remark made in §7 of the *Theory of Science*:

... in the examples which are used in logic, e.g., in the syllogism: all A are B , all B are C ; therefore all A are C , the signs A, B , and C may mean, as we say, "anything". But this statement is not altogether precise. The signs A, B , and C can here mean very different things, but not quite anything we may choose. They must signify ideas such that B is an idea which can be predicated of all A and C one which can be predicated of all B . Thus it can

¹ One could also make sense of this case by beginning with a different proposition, say $[1=2]$, passing to the form ' $a=b$ ', and thence to $[2=2]$, in which case we wouldn't have to use non-uniform substitutions. That Bolzano permits us to substitute the same idea for both a and b (and thus consider $[2=2]$ as belonging to the form ' $a=b$ ') is clear from, e.g., **WL**, §228 [II.416].

² We will in the following always assume that substitutions are uniform.

³ *Logical Investigations*, IV, §10.

be seen that the objects *A*, *B*, and *C* are not left indeterminate as to all their characteristics, but only as to some of them.¹

We see another indication of his awareness of the necessity of some such constraint in a remark from §81:

... when the letter *A* represents any subject-idea, and the letter *b* any attribute-idea, then the expression “*A* has *b*” is the general form of any proposition whatever...²

Based on such examples, one might suppose that for Bolzano a form is only fully specified once we have indicated which sorts of ideas may be substituted for those declared variable. Apart from occasional remarks like those just mentioned, however, he says very little on this subject.^{3,4}

(C) ANALYTICITY AND RELATED NOTIONS

With these somewhat protracted discussions behind us, let us return to Bolzano’s treatment of the concept of analyticity. We have already seen that for Bolzano analytic propositions are those for which all the objectual variants

¹ WL, §7, no. 5 [I.28].

² WL, §81, note 2 [I.393].

³ Wolfgang Kühne, “Analyticity and logical truth: from Bolzano to Quine,” has an excellent discussion of these and related problems, as does M. Siebel, *Der Begriff der Ableitbarkeit bei Bolzano*, pp. 73 ff.

⁴ Alberto Coffa (*The Semantic Tradition from Kant to Carnap: To the Vienna Station* (Cambridge: Cambridge University Press, 1992), p. 378, note 9) suggested that Bolzano intended the requirement that objectuality be preserved to filter out substitutions across semantic categories. Coffa—perhaps following Quine—actually speaks of “grammatical admissibility”. Kühne (“Analyticity and logical truth: from Bolzano to Quine,” p. 232, note 22) calls this a “serious misunderstanding”, and it is hard to disagree with him. For one thing, as Kühne points out, grammatical distinctions lie at the level of language rather than at the level of propositions in themselves. For another (again noted by Kühne), there are grammatically admissible substitutions which result in non-objectual propositions (e.g., substituting [W. V. O. Quine] for [Noam Chomsky] in the proposition [Noam Chomsky, the MIT professor, is a hero of the counterculture.]). A third problem is that the objectuality requirement only covers the subject-idea of a proposition, so even if some cases of categorially inappropriate substitutions within the subject-idea were filtered out by this means, none of those which only affected the predicate-idea would be. On balance, then, it seems safe to say that the requirements of objectuality and the preservation of semantic category (which Bolzano mentions far less often) are distinct.

(relative to some selection of variable parts) have the same truth value. Obviously, this does not occur in most cases. Rather, we generally find that some variants of a given proposition are true, while others are false. Beginning, for example, with a proposition such as

[The human being Solon is wise]

and treating [Solon] as the only variable part, we will find that some substitutions produce true propositions, others false ones (even when the subject-idea remains objectual).

Suppose now that we have a set of ideas which contains, for each human being, exactly one idea representing it and it alone, and nothing else besides. In this case, the ratio of the number of ideas in this set which produce true propositions when substituted for [Solon] to the total number of ideas in the set will then represent the proportion of human beings who are wise. It will also equal the probability that a randomly selected human being will be wise. Thus if the set of allowable substitutions is specified in the right way, we will obtain a measure of absolute probability. Bolzano calls this the *degree of validity* of a proposition with respect to given variable parts.¹

In this way Bolzano obtains the same result as would be obtained with object variation and propositional functions, because the set of allowable substitutions may be mapped one-to-one onto the set of objects in question. Note too that the specification of an appropriate substitution-class is part of the definition of the degree of validity:

The degree of validity of a proposition is expressed by the relation between the number of the true propositions and the total number of propositions which are generated when certain ideas contained in the original proposition are considered variable and exchanged for others *according to a certain rule*.²

Finally, Bolzano takes care to point out that the degree of validity of a proposition belongs to it not absolutely, but relatively—relative, namely, to a collection of variable parts as well as to the rule governing allowable substitutions or, what amounts to the same thing, relative to one of its forms.³

Analytic propositions appear as special, limit cases of the above relation. If the degree of validity is one, then all permissible substitutions result in true

¹ WL, §147.

² WL, §147 [II.81], emphasis added.

³ WL, §147.

propositions. If it is zero, all permissible substitutions produce false propositions. Bolzano calls the former type of propositions *universally valid*, the latter *universally invalid*, relative to the given variable parts. A proposition which is either universally valid or universally invalid with respect to some collection of variable parts is said to be *analytic*; the remaining propositions are called *synthetic*. Analytic propositions, then, are either fully probable or fully improbable with respect to the given variables. They can also be said to be *true (or false) after their kind*, in the sense that every objectual proposition of the same kind (i.e., belonging to the same form) has the same truth value—though we should take care to note that each proposition belongs to several different forms, only some of which may be universally valid. The proposition

[The man Caius is mortal]

for example, belongs to the universally valid form

The man X is mortal

but also to form

The man Caius is A .

While the proposition

$[2 < 3 \text{ and } 2 = 3]$

belongs to the universally invalid form

$x < y \text{ and } x = y$

though it is not universally invalid if “=” is the only variable idea.

Thus in order to express what Bolzano had in mind, we should say that an analytic proposition is one which is either true or false after *at least one* of its kinds.

Here are some of the examples of analytic propositions (or universally valid/invalid propositional forms) that Bolzano cites (the parts with respect to which the proposition is claimed to be universally valid or invalid are indicated either by underlining or by letters):

1. [The man Caius is mortal.] (WL, §147)
2. [The man Caius is omniscient.] (WL, §147)
3. [A depraved man does not deserve respect.] (WL, §148)

4. [A man may be depraved and still enjoy continued happiness.] (WL, §148)
5. [A is A.] (WL, §148)
6. [An A which is a B is an A.] (WL, §148)
7. [An A which is a B is a B.] (WL, §148)
8. [Every object is either B or non-B.] (WL, §148)
9. [If all men are mortal and Caius is a man, then Caius is mortal.] (WL, §315)
10. [If A is larger than B, then B is smaller than A.] (WL, §148)
11. [If $P = Mm$, then $M = \frac{P}{m}$.] (WL, §148)
12. [$a + (b + c) = (a + b) + c$] (WL, §315)
13. [The soul of Socrates has been annihilated.] (WL, §369)
14. [The soul of Socrates is a simple substance.] (WL, §447)
15. [If $\frac{a^2}{2} = b$, then $a = \pm\sqrt{2b}$] (WL, §447)

Logical Analyticity

Bolzano introduces the concept of *logical* analyticity in §148 of *Theory of Science* by contrasting the analyticity of propositions such as

[Every object is either red or non-red]

with that of propositions such as

[A depraved man does not deserve respect.]

He remarks that in order to appreciate the analyticity of the former, “only logical knowledge is needed, since the concepts which form the invariable part of these propositions all belong to logic,” while with the latter “a wholly different kind of knowledge is required, since concepts alien to logic intrude.” To this he adds:

This distinction, I admit, is rather unstable, as the whole domain of concepts belonging to logic is not circumscribed to the extent that controversies could not arise at times. Nevertheless, it might be profitable to keep this distinction in mind. Hence propositions like [the former] may be called *logically* analytic, or analytic

in the narrower sense; [and the latter], analytic in the *broader* sense.¹

Although Bolzano mentions an epistemological (only logical *knowledge* is required) as well as a logical criterion (the only invariable parts are logical concepts), the fact that the concept is introduced in the *Theory of Elements*, which deals with propositions and ideas *in themselves* (i.e., without reference to thoughts or linguistic expressions), weighs heavily in favour of the view that the fundamental distinction is the logical one. On this reading, the fact that the only invariable parts are logical concepts is supposed to explain why only logical knowledge is required.²

Tarski, in his 1936 paper on logical consequence, had defined analytic sentences as those which are consequences of every set of sentences and hence in particular of the empty set; or, what amounts to the same, those which are true on every interpretation.³ Recall that on Tarski's understanding of substitution (or rather, the formation of sentential functions) we divide the *vocabulary* of a language into two parts, logical and non-logical, and then declare (uniformly) variable *all and only* the non-logical parts. Bolzano does not say this, though he has been interpreted in this way.⁴ Rather, he says that a proposition is logically analytic if universally valid relative to a form where the only invariable parts are logical concepts. The difference is significant—Tarski's definition identifies analyticity with the universal validity of a uniquely determined form (which one might call the *ultimate* or *finest* logical form of a sentence⁵). Bolzano's, by contrast, does not. Consider, for example, the proposition:

¹ WL, §148, no. 3 [II.84].

² For further discussion of this point, see M. Textor, "Bolzano et Husserl sur l'analyticité," *Études Philosophiques* 4 (2000) 435–54; W. Künne, "Analyticity and logical truth: from Bolzano to Quine," pp. 184–249 in M. Textor, ed., *The Austrian Contribution to Analytic Philosophy* (London and New York: Routledge, 2006), pp. 203 and 238, note 71.

³ A. Tarski, *Logic Semantics, Metamathematics*, 2nd edn (Indianapolis: Hackett, 1983), pp. 417–18. Tarski here uses Carnap's terminology from the *Logical Syntax of Language*. Note that "analytic" in their sense only covers universally valid propositions, while in Bolzano's sense it covers universally invalid propositions as well. The latter are called *contradictory* by Carnap and Tarski.

⁴ e.g., by Y. Bar-Hillel, "Bolzano's definition of analytic propositions," *Methodos* 2 (1950) 32–55, p. 41; J. Berg, editor's introduction to *BBGA*, I.12/1, p. 18.

⁵ Relative to a prior specification of the logical vocabulary of the language in question.

[Either 2 is prime and 17 is even or it is not the case that 2 is prime and 17 is even.]

For Bolzano, this proposition belongs to many forms, several of which only number logical concepts among their invariable parts, e.g.:

$$A \vee B$$

$$A \vee \neg A$$

$$(A \wedge B) \vee \neg(A \wedge B)$$

$$(Fa \wedge Gb) \vee \neg(Fa \wedge Gb)$$

For Tarski, by contrast, only the last of these forms comes into consideration. We see, then, that by Bolzano's lights a proposition may belong to several purely logical forms, and indeed may be universally valid with respect to some of them but not others.¹

The advantages of Bolzano's definition can be seen when we consider that, for him, verbal expressions were rarely perfectly distinct—i.e., did not always accurately reflect the ultimate components of the propositions and ideas in themselves which they designate—and that, moreover, we are in many cases unsure of what those components are. In cases of doubt, we would be unable to determine the ultimate, "Tarskian" form of a proposition and hence unable to determine directly whether it was logically analytic in Tarski's sense or not; while on Bolzano's view, we might still have enough knowledge about the structure of such a proposition to be able to recognize that it is logically analytic. Bolzano, for example, gave perhaps the first precise definition of the continuity of a function on an interval. Before then, however, mathematicians had used the concept. Before knowing the definition, the ultimate logical form of the following proposition would be unknown to us:

[A function f is either continuous on a given interval or it isn't.]

But since we know that there are functions and can see that this proposition is of the form

An A is either B or not B

we can, on Bolzano's account, still recognize that it is logically analytic.

¹ Cf. W. V. O. Quine, "Grammar, truth, and logic," and D. Føllesdal, "Comments on Quine," both in S. Kanger and S. Öhman, eds, *Philosophy and Grammar* (Dordrecht: Reidel, 1980), pp. 17–28, 29–35.

The above difference is also important in light of Bolzano's view that the boundary between logical and non-logical concepts is unstable [*schwankend*]. For even if we are not sure that there is a sharp boundary, or (supposing there is one) exactly where it lies, and hence exactly what the ultimate "Tarskian" logical form of a given proposition is, we may still be in a position to recognize that it is logically analytic on Bolzano's account.

Suppose, for example, that [A] is a proposition containing several concepts that, as far as we can tell, might or might not be logical. In this case, we cannot say with confidence what the ultimate logical form of this proposition is. Even so, we can still say, on Bolzano's account, that the proposition [A or not A] is logically analytic, provided we are confident that [or] and [not] are both logical concepts.

This being said, if a proposition is universally valid (or invalid) with respect to one purely logical form, it is easy to see that it must also be universally valid (or invalid) with respect to its ultimate, "Tarskian" form. For any finer variations within the parts considered variable in the "grosser" form will simply produce ideas which were already in the range of possible substitutions. In this case, we could be confident in the claim that a proposition was universally valid (or invalid) with respect to its ultimate logical form (again supposing that there is such a thing) even if we didn't know what that form was.

(D) CONSEQUENCE AND RELATED NOTIONS

We now turn to the relations between *several* propositions which come to light when we consider some of their constituents variable, among them compatibility (consistency or joint satisfiability), deducibility (consequence), equivalence, and probability.

Bolzano arrives at his definitions of these relations via a deep analogy between the logic of classes and the logic of propositions. The key to the analogy is to see the correspondence between an object standing under an idea and a substitution making a proposition true.

With ideas, the crucial question was whether or not a certain object is indeed represented by them; the corresponding question for propositions is whether or not they are true. Just as I have called ideas compatible or incompatible with each other, depending upon whether or not they have certain objects in common, so I call propositions compatible or incompatible, depending upon whether or not there are certain ideas which make all of them true.¹

¹ WL, §154, no. 4 [II.101].

In virtue of this correspondence, one can apply the schema provided by the logic of classes directly to the logic of propositions, arriving at a complete system in one decisive step.

To illustrate: consider the propositions

- (1) [Canada is in the northern hemisphere.]
- (2) [Brazil is in the northern hemisphere.]

If we consider only the idea [northern] variable, we obtain the forms

- (1') Canada is in the A hemisphere.
- (2') Brazil is in the A hemisphere.

Now there is a class of appropriate substitutions for A, including such ideas as [northern], [southern], [eastern], etc. Note, in particular, that the substitution [western] results in the two true propositions:

- (1'') [Canada is in the western hemisphere.]
- (2'') [Brazil is in the western hemisphere.]

That is, the class of substitutions which make (1') true and the class of substitutions which make (2') true have a member in common, i.e., the *ideas*

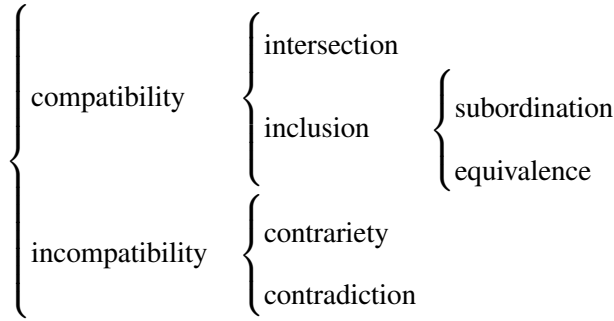
[idea which, when substituted for [northern] in the first proposition, results in a truth]; and

[idea which, when substituted for [northern] in the second proposition, results in a truth]

are *compatible* (see above, p. 219). Bolzano will accordingly say that the original *propositions* (1) and (2) are compatible with respect to the variable idea [northern], and similarly for the other relations between ideas.

Making use of modern terminology and symbolism, we can give a compact overview of Bolzano's system of relations between propositions.

Let $\mathcal{A} = \{A, B, C, \dots\}$ and $\mathcal{M} = \{M, N, O, \dots\}$ be sets of propositions, and suppose that ideas $i, j, k, \dots$ occurring in these propositions are considered variable. Let $\mathcal{U}$ designate the set of all appropriate substitutions for $i, j, k, \dots$ and $\emptyset$ the empty set. Let $T(\mathcal{A})$ denote the set of substitutions which make all of $A, B, C, \dots$ true, and similarly for $T(\mathcal{M})$. Recalling now the classification of relations between the extensions of ideas:



We can define analogous relations between propositions with variable parts (i.e., between propositional forms). Then we will say that

- $\mathcal{A}$ and $\mathcal{M}$ are *compatible* iff $T(\mathcal{A}) \cap T(\mathcal{M}) \neq \emptyset$
- They are *incompatible* iff $T(\mathcal{A}) \cap T(\mathcal{M}) = \emptyset$
- $\mathcal{M}$ are *deducible* from $\mathcal{A}$ (or $\mathcal{A}$ are *included* in $\mathcal{M}$) iff $T(\mathcal{A}) \neq \emptyset$ and $T(\mathcal{A}) \subseteq T(\mathcal{M})$
- $\mathcal{A}$ and $\mathcal{M}$ are *equivalent* iff $T(\mathcal{A}) \neq \emptyset$ and $T(\mathcal{A}) = T(\mathcal{M})$.

The other relations may be defined similarly. Finally, let us note something that Bolzano didn't take the trouble to point out, namely, that if A is a single proposition with variable parts $i, j, k, \dots$ and $\mathcal{U}$ the set of appropriate substitutions (here *appropriate* also means that the substitutions result in *objectual* propositions), then

- A is universally valid with respect to $i, j, k, \dots$ iff $T(A) = \mathcal{U}$
- A is universally invalid with respect to $i, j, k, \dots$ iff $T(A) = \emptyset$.

By means of this far-reaching analogy, the theorems of the theory of classes can also be immediately applied to the logic of propositions.

Let us now take a closer look at Bolzano's presentation.

Compatibility

A set of *ideas* was said to be compatible if there was at least one object represented by all of them. Analogously, a set of *propositions* is said to be compatible (as always relative to certain variands) if there is at least one substitution that makes all of them true:

Let us compare several propositions $A, B, C, D, \dots$ and consider as variable certain ideas $i, j, \dots$, which they have in common. The question arises whether there are any ideas which can be put into the place of $i, j, \dots$ and which are of such a nature that they make all of the above propositions true at the same time. If this question must be answered in the affirmative, then I wish to call the relation between propositions $A, B, C, D, \dots$ a relation of *compatibility*, and the propositions $A, B, C, D, \dots$ will be called *compatible* propositions.¹

The propositions [6 is a square number] and [6 is an odd number], for instance, are compatible (where [6] is the only variable), since the substitution of [25] makes both propositions true, while [The figure ABC is a triangle] and [two of the angles of figure ABC are right angles] are incompatible (where ABC is the only variable).

We have already seen that the same proposition may be universally valid with respect to certain variables not with respect to others. The same thing occurs with compatibility, and indeed with all of Bolzano's relations between propositions. Thus, for example, the propositions

[Bolzano died before Příhonský]

[Příhonský died before Bolzano]

are incompatible if the occurrences of [Bolzano] and [Příhonský] are the only ideas that are considered variable, but compatible when the two occurrences of [died before] are taken as the variable parts (since, for example, substituting [respected] for [died before] results in two true propositions).

We will mention two important theorems which Bolzano proves in §154:

1. If a set of propositions $\{A, B, C, \dots\}$ is compatible w.r.t. certain variables, then so is any non-empty subset of $\{A, B, C, \dots\}$.²
2. If a set of propositions is compatible with respect to certain variables $i, j, k, \dots$, then it is also compatible w.r.t. any set of variands that contains $i, j, k, \dots$ as a subset.³

¹ WL, §155, no. 2.

² WL, §154, no. 10. Bolzano there establishes the contrapositive.

³ WL, §154, no. 11.

Deducibility and Equivalence

We now turn to the central concept of any logical system, namely, consequence, which Bolzano calls *deducibility* (*Ableitbarkeit*). Deducibility between propositions corresponds to the relation of inclusion between ideas, and, like it, is defined as a special case of compatibility:

Let us consider, first of all, the case that among the compatible propositions $A, B, C, D, \dots, M, N, O, \dots$ the following relation obtains: all ideas whose substitution for the variable ideas $i, j, \dots$ turns a certain part of these propositions, namely, $A, B, C, D, \dots$, into truths, also have the characteristic of making a certain other part of these propositions, namely, $M, N, O, \dots$ true. [...] I wish to give the name of *deducibility* to this relation between propositions $A, B, C, D, \dots$ on one hand and $M, N, O, \dots$ on the other. [...] I say that propositions $M, N, O, \dots$ are *deducible* from propositions $A, B, C, D, \dots$ with respect to variable parts $i, j, \dots$, if every collection of ideas whose substitution for $i, j, \dots$ makes all of $A, B, C, D, \dots$ true, also makes all of $M, N, O, \dots$ true.¹

Bolzano's notion of deducibility differs from modern notions of logical consequence in several important respects. First of all, it is not a notion of *logical* consequence, but rather a broader concept which covers logical consequence as a special case (namely, the one where all invariable parts are logical concepts²). Thus not only is [Socrates is mortal] deducible from the premises [Socrates is human] and [All humans are mortal], but also [16 – 9 is an odd number] is deducible from [16 and 9 are consecutive square numbers], since every substitution for [16] and [9] that makes the latter true also makes the former true. Similarly, [Pele is in Brazil] is deducible from [Pele is in Rio de Janeiro] with respect to [Pele].

Second, the compatibility requirement complicates life somewhat.³ For example, deducibility is not a reflexive relation in general, since if A is to be deducible from itself with respect to a given class of variables, at least one substitution for these variable parts must make it true. This is not the case, for example, with the proposition [Aristotle is both mortal and not mortal],

¹ WL, §155, no. 2 [II.113–14].

² WL, §223.

³ We shall see below (p. 315) that Bolzano had at least one good reason for including this feature.

when [Aristotle] and [mortal] are considered variable. For similar reasons, contraposition is not generally valid in Bolzano's system. For even if B is deducible from A with respect to variables $i, j, \dots$, it might happen that $\text{not-}B$ is not transformed into a true proposition by any substitution for $i, j, \dots$. In this case, $\text{Not-}A$ is not deducible from $\text{Not-}B$ (example: [Aristotle is mortal], [Aristotle is either mortal or he isn't]; variables: [Aristotle], [mortal]).

Third, the set of parts to be considered variable is not fixed once and for all at the beginning, but rather is itself allowed to change from one logical situation to another—for example, from one step of an argument to the next. Deducibility is thus a *triadic* relation involving collections of premises, conclusions, and variable parts as its terms.

One important consequence of this feature is that deducibility (with respect to some set of variable parts or another) is not transitive in general; that is, we may have a proposition C deducible from another B , and B in turn from A , without C being deducible from A .¹ Here is a simple example. [Xanthippe is mortal] is deducible from [Xanthippe is human] with respect to [Xanthippe], and [Socrates' wife is mortal] is deducible from [Xanthippe is mortal] with respect to [mortal], but [Socrates' wife is mortal] is not deducible from [Xanthippe is human] with respect to any non-trivial selection of variable parts.²

This is a little easier to see when presented in the following format (where the boxes indicate the variable parts):

Xanthippe	is human.	Xanthippe	is	mortal	.
Xanthippe	is mortal.	Socrates' wife	is	mortal	.
Xanthippe is human. Socrates' wife is mortal.					

Clearly, any substitution for the boxed items in the first two inferences which makes the premise true will also make the conclusion true. Equally clearly, no selection of variable parts yields a relation of deducibility in the third.

¹ Of course, transitivity does hold when the set of variable parts remains the same throughout. Bolzano proves this, and a related theorem for cases where variables are not the same, and hence additional conditions are required, in nos. 24 and 25 of **WL**, §155.

² For an example that also works in all cases, substitute [snake] for [human] and [reptile] for [mortal].

One moral that can be drawn from the failure of transitivity is that we must distinguish *provability* from *deducibility* in Bolzano's system. As we have just seen, for example, from the premise [Xanthippe is human] we can prove (i.e., establish by means of a sequence of correct inferences) [Socrates' wife is mortal], even though the latter is not deducible from the former.

The existence of the additional parameter (the parts that are considered variable in a given case of deducibility) also allows us to make sense of proofs by reduction to absurdity in Bolzano's system. On the face of things, one might doubt that such proofs can be carried out. Mark Siebel, for example, has written:

[T]he condition of compatibility does not allow for *reductio ad absurdum*. In such inferences, a contradiction is derived from the assumptions in order to prove that they are inconsistent. For Bolzano, this procedure cannot even get going because there is nothing which is derivable from incompatible propositions, be it a contradiction or something else. This is odd because Bolzano himself (e.g., in WL II, §155) uses *reductio ad absurdum*.¹

This observation is certainly true if the same variables are used throughout. But they don't have to be, as the following example shows.

Let B = [Bush won the 1992 Presidential election] and C = [Clinton won], and suppose we wish to prove $[\neg(B \wedge C)]$ indirectly from the premise $[\neg B]$. Adding $[B \wedge C]$ as a *reductio* assumption, we can then deduce the following conclusions (where the variable parts are indicated by the boxes, and the corresponding inference schemes appear on the right)

$\boxed{\neg B}$	α
$\boxed{B \wedge C}$	$\beta \wedge \gamma$
$\boxed{B}$	β

i.e., $[B]$ is deducible from the augmented set of premises which includes the assumption.

¹ M. Siebel, "Bolzano's Concept of Consequence," *Monist* **85** (2002) 580–99, p. 586; cf. *Der Begriff der Ableitbarkeit bei Bolzano* (St Augustin: Academia, 1996), pp. 109 f.

Next:

$\boxed{\neg B}$	α
$\boxed{B \wedge C}$	β
$\boxed{\neg B}$	α

i.e., $[\neg B]$ is also deducible from the augmented set of premises.

Note that this does not show that the premises in the augmented set are incompatible for, clearly, they are compatible when the variables are as indicated above. What the deduction of both $[B]$ and $[\neg B]$ does show, however, is that not all the conclusions deducible from the augmented set of premises are *true*. And since all conclusions which are deducible from true premises are themselves true, we can conclude that either the premise $[\neg B]$ or the *reductio* assumption $[B \wedge C]$ is false. Since our premise is true, the assumption must be false, and hence its negation, i.e., $[\neg(B \wedge C)]$, true. This completes the proof. Other indirect proofs can obviously be carried out along similar lines.

This being said, there are some cases where an indirect proof is possible in modern systems of logic, but not in Bolzano's. For example, it isn't possible to prove $[C]$ from the premise $[B \wedge \neg B]$ in Bolzano's system, even though we can deduce both $[B]$ and $[\neg B]$ from the premises $[B \wedge \neg B]$ and $[\neg C]$. For while this allows us to conclude that not all of the premises used in the deduction are true, as in the above case, we cannot then proceed to conclude that the *reductio* assumption must be false.

The deducibility of a proposition M from $A, B, C, \dots$ relative to variables $i, j, k, \dots$ entails, and is entailed by, the incompatibility of $A, B, C, \dots$ with the negation of M (with the proviso that $A, B, C, \dots$ are compatible). Because of this, we have the following two theorems corresponding to the theorems on compatibility cited above (p. 304):

1. If M is deducible from $A, B, C, \dots$ relative to variables $i, j, k, \dots$, then M is also deducible from any compatible set of propositions containing $A, B, C, \dots$ w.r.t. to the same variables.
2. If M is deducible from $A, B, C, \dots$ relative to variables $i, j, k, \dots$, then M is also deducible from $A, B, C, \dots$ relative to any non-empty subset of $i, j, k, \dots$, provided that $A, B, C, \dots$ are compatible relative to the smaller set of variables.

There are many others. Indeed, it does not seem exaggerated to say that the collection of theorems in this part of the *Theory of Science* is the most substantial in the history of logic up to that point.

Equivalence is defined as mutual deducibility.¹ Equivalence is reflexive, symmetric, and transitive. The other key feature of this relation is that equivalent sets of propositions may be exchanged for each other without altering logical relations. That is, if $A, B, C, \dots$ are equivalent to $A', B', C', \dots$ with respect to variables $i, j, k, \dots$ and $M, N, O, \dots$ are either compatible/incompatible with, deducible from, or equivalent with $A', B', C', \dots$, with respect to the same variables, then the same relation also holds between $M, N, O, \dots$ and the equivalent propositions $A, B, C, \dots$.²

The concept defined above is deducibility in the broad sense, a triadic relation involving premises, conclusions, and a selection of variables. As was the case with analyticity, we may also speak of a special variety of deducibility which he calls *logical deducibility*.³ The former concept is not to be met with in most contemporary presentations of logic, which deal only with the narrower concept of *logical consequence* (in Tarski's sense), based upon a previous separation of the logical from the non-logical terms. On Bolzano's understanding, the relevant selection of variables is not fixed once and for all at the beginning, but rather can shift from one context of argumentation to the next, and even within a particular chain of arguments. While it may be true, for example, that [Socrates is a primate] does not follow from [Socrates is a man], when all occurrences of [Socrates], [primate], and [man] are declared uniformly variable, since there are substituends which will render [X is a B] true and [X is a C] false, we can by no means conclude that this argument is invalid, or that it must involve a tacit premise, viz. [All men are primates.] For the argument clearly has a valid form, namely

X is a man. So X is a primate.

It follows that, in order to assess arguments, we need to know not only what their premises and conclusions are, but also which choice of variables is intended.⁴

Contemporary treatments of logic do not deal with this more general concept (or the corresponding more general concepts of compatibility, etc.).

¹ WL, §156.

² WL, §156, no. 4 [II.134].

³ Cf. WL, §223; ML, §8, no. 3 [BBGA 2A.7, p. 64; MM-EX, p. 55].

⁴ WL, §223. Cf. R. George, "Bolzano's consequence, relevance and enthymemes, with a postscript on fallacies," *Journal of Philosophical Logic* 12 (1983) 299–325.

where some of the non-logical elements remain invariable. It is nevertheless quite useful, since it permits us to make sense of more specialized, or “regional” logics of particular disciplines. The very same logical operations are thus used, but with a richer stock of invariable concepts. In mathematics, for example, one might consider concepts of operators such as $[+]$, $[-]$, $[\times]$, $[\div]$, etc., as well as concepts of relations like $[=]$, $[>]$, and so on invariable, allowing only ideas of numbers to vary. In this case, there is no need to appeal to “tacit premises” and the like in order to justify the usual mathematical inferences. Previous logicians had spoken of general and special logics, but in Bolzano’s system it becomes possible to associate a precise and fruitful sense with these expressions.

Interestingly, to claim that the argument form

X is a man. So X is a primate

is valid on Bolzano’s definition (i.e., that there is a genuine case of consequence here) amounts to saying that every substitution for *X* that makes the first propositional form true also makes the second true or, equivalently, that all men are primates, which is precisely the premise declared “tacit” or “missing” in the usual treatments of enthymemes. As Rolf George has shown, the same holds for any enthymeme obtained by deleting a premise from a valid classical categorical syllogism.¹ Bolzano’s understanding of consequence thus yields an elegant account of classical enthymemes.

Bolzano’s definition covers many if not all of the cases of logical consequence recognized by earlier as well as later logicians. The syllogism

All mammals are vertebrates. Some mammals do not have eyes.
So not all vertebrates have eyes

for example, can be declared valid, because it has the valid form

All *A* are *B*. Some *A* are not *C*. So Not all *B* are *C*.

So too, the arguments:

(1) There is a natural number that is less than or equal to all natural numbers. So for every natural number, there is a natural number less than or equal to it.

¹ R. George, “Bolzano’s consequence, relevance, and enthymemes.”

(2) Napoleon had all the qualities of a great general. Foresight is one of the qualities of a great general. So Napoleon had foresight.

(3) Cicero was a great orator. Cicero was none other than Tully. So Tully was a great orator.

can be declared valid by virtue of the forms (where the letters, apart from those representing bound variables, indicate the variable parts,¹ etc.):

$$(1') \exists x \forall y Rxy \therefore \forall y \exists x Rxy$$

$$(2') \forall P(\forall x(Gx \rightarrow Px) \rightarrow Pn), \forall x(Gx \rightarrow Fx) \therefore Fn$$

$$(3') Oc, c = t \therefore Ot$$

Before Bolzano, only Leibniz had come close to a concept of deducibility that went beyond the usual definitions based upon the syllogistic form (Leibniz's text was only published in 1903 by Couturat). According to Leibniz, an inference is valid if it "is conclusive in virtue of its form, that is, it must always succeed when one substitutes arbitrary instances for the present one."² Bolzano goes farther: he fixes the variables as well as the collections of ideas that are destined to replace them, defines logical deducibility as the special case where only logical ideas remain invariable, and integrates his concept in a coherent and comprehensive set of extensional relations between propositions.

Exact Deducibility

In §155.26 of the *Theory of Science*, Bolzano distinguishes a special case of deducibility, which he defines as follows:

¹ In the usual semantics, where the domain of quantification is also allowed to vary, one must also think of the quantifiers as containing a tacit parameter, representing a non-empty set. In this case, 'For all x ' should be understood to mean 'For all x in S '.

² Leibniz, *Opusculæ et fragments inédits* (Paris: Alcan, 1903), 338–9: *Mihi verò omnis ratiocinatio quæ vi formæ concludit, hoc est quæ semper successura est, substitutis in præsentis exempli locum exemplis aliis quibuscunque, rectam formam habere videtur.* Cf. *New Essays*, IV, xvii.

Let proposition M be deducible from premises $A, B, C, D, \dots$ with respect to ideas $i, j, \dots$. If $A, B, C, D, \dots$ are such that none of them, nor even any of their parts, may be omitted, with M still deducible from the remainder with respect to the same ideas $i, j, \dots$, I call the relation of deducibility of proposition M from $A, B, C, D, \dots$ *exact, precise, or adequate*.

In the essay on mathematical method, he gives a different definition:

In a *narrower* sense—the one in which I will henceforth understand this way of speaking—I say that a proposition M is *deducible* from the propositions $A, B, C, \dots$ if each collection of ideas that, when substituted for $i, j, \dots$ makes all of $A, B, C, \dots$ true also makes the proposition M true, and when the same does not hold for any *part* of the [collection of] propositions $A, B, C, \dots$ —i.e., if it is not also the case that whenever only a part of this [collection of] propositions becomes true, M does as well.¹

From the theorems he states in the *Theory of Science*, his intent is clear: he is looking for a narrower concept of deducibility, where no idle elements are involved. Unfortunately, neither of his definitions yields quite what he wanted.

Using the second definition, for instance, we would have to say that an argument form like

$$\frac{(P \rightarrow Q) \wedge R}{P} \\ \hline Q$$

was a case of exact deducibility, despite the presence of the idle conjunct R in the first premise. Thus the second definition seems too wide. Perhaps this is why in the *Theory of Science* he had added the clause “or even any of their parts.” By omitting the part $[\wedge R]$ from the first premise, we would still have a relation of deducibility, namely:

$$\frac{P \rightarrow Q}{P} \\ \hline Q$$

¹ **ML**, §8, no. 2 [**BGA** 2A.7, p. 63; **MM-EX**, p. 54].

So that on the first definition, the above deduction would not be exact. Yet in other cases, deleting parts causes trouble of a kind Bolzano did not want. Consider a dilemma, e.g.:

$$\begin{array}{c} P \vee Q \\ P \rightarrow R \\ \hline Q \rightarrow R \\ \hline R \end{array}$$

According to the second definition, this deduction is exact—which seems correct—but according to the first definition, it is not; for, omitting the part $[\vee Q]$ from the first premise, we obtain

$$\begin{array}{c} P \\ P \rightarrow R \\ \hline Q \rightarrow R \\ \hline R \end{array}$$

an argument which is not only valid, but even inexact.

With the second definition one can prove some important results. To begin with, in cases of exact consequence, none of the premises, nor the conclusion may be universally valid with respect to the variables of the deduction. For if one of the premises had that character, it could be deleted without invalidating the consequence, while a universally valid conclusion, for its part, “does not need for its truth the condition that the premises are true.”¹ Hence in neither case would the consequence be exact. We also have the following: If M follows exactly from $A, B, C, \dots$ with respect to some variands, then its negation $\text{Neg}.M$ is compatible with any proper subset of the premisses. For if $\text{Neg}.M$ were incompatible with any such subset, then $\text{Neg}.\text{Neg}.M$ would follow from it, hence M would follow, and the consequence would not be exact. It follows further that in such a relation no premise can be a consequence of the rest, and the negation of any of them must be compatible with the rest. In other words, the premises of an exact argument form must be independent of one another.

Another highly interesting feature of exact deducibility is the following: If a proposition M is exactly deducible from $A, B, C, \dots$ relative to certain variables $i, j, k, \dots$, then at least one of these variables must occur in one of

¹ WL §155, no. 27.

the premises as well as in the conclusion. For suppose that M is deducible from $A, B, C, \dots$ with respect to the variable parts $i, j, k, l, \dots$, but that none of $i, j, \dots$ occur in M and none of $k, l, \dots$ occur in any of $A, B, C, \dots$. Since $A, B, C, \dots$ are compatible, some substitution of ideas for $i, j, \dots$ makes all of them true. But since M is deducible from $A, B, C, \dots$, it must also be true whenever they are. Leaving the above substitutions for $i, j, \dots$ fixed, it is clear that any substitution for $k, l, \dots$ in M must make it true. Thus M is universally valid with respect to $k, l, \dots$ and hence, since none of $i, j, \dots$ occur in M , also with respect to $i, j, k, l, \dots$. Thus M is not exactly deducible from $A, B, C, \dots$. Exact deducibility thus incorporates a criterion of *relevance*.¹

NOTE

In an article of 1983 (“Bolzano’s consequence, relevance, and enthymemes,” *Journal of Philosophical Logic* **12** (1983) 299–318; pp. 310 ff.), Rolf George argued that the above condition of relevance should be recognized as a feature of Bolzano’s concept of deducibility. Mark Siebel (“Bolzano’s concept of consequence,” *Monist* **85** (2002) 580–99; pp. 594 ff.; cf. *Der Begriff der Ableitbarkeit bei Bolzano* (St Augustin: Akademie Verlag, 1996), pp. 225 ff.) later challenged this, noting that on Bolzano’s general concept of deducibility, there are cases of deducibility where premises and conclusion do not share any variables (e.g., [2 is an even number] is deducible from [The Eiffel Tower is in France] relative to the variable [France], since any substitution for [France] which makes the second proposition true also (trivially) makes the first proposition true.) Yet George had not claimed that the shared variable property could be proved of Bolzano’s concept of deducibility alone. Rather, he claimed (and showed) that it could be proved if we added a postulate stating that a proposition is deducible from certain others with respect to given variables, only if it is not universally valid with respect to them (George, Postulate 2; p. 309; in his paper “Variation, derivability, and necessity,” *Grazer phil. St.* **53** (1997) 117–37, p. 130, Siebel acknowledges this). Now this postulate is a theorem, as we have just noted, not about Bolzano’s general relation of deducibility, but rather about his relation of exact deducibility. Siebel also claims (p. 596 of the *Monist* article and 1997, p. 130) that “Bolzano offers the condition of shared variable ideas as if it was a *theorem* of his definition. . . .” (Rusnock and George also made this claim in the article “Bolzano as Logician,” in D. Gabbay and J. Woods, eds, *Handbook of the History and Philosophy of Logic*, Vol. 3 (Amsterdam: Elsevier, 2004), pp. 177–205, p. 197; obviously, Rolf and I (PR) weren’t paying close enough attention at that time.) It seems to us that this claim, which is based upon Bolzano’s remarks in **WL**, §155.21, is incorrect. At that place, Bolzano argues that it is not always possible, by a judicious selection of variables, to place *any* given propositions into a relation of deducibility; this result is

¹ Cf. W. Stelzner, “Compatibility and Relevance: Bolzano and Orlov,” *Logic and Logical Philosophy* **10** (2002) 137–71.

the counterpart to §154.5, where Bolzano claims that it is possible to place any given propositions in a relation of *compatibility*. He argues as follows [II.120]:

It is not the case that any proposition M , let alone any arbitrary collection of propositions $M, N, O, \dots$ can be placed into a relation of deducibility with another individual proposition A or a collection of propositions $A, B, C, D, \dots$ simply by stipulating that certain of their ideas $i, j, \dots$ should be considered variable. Given, for example, that the two propositions “ A has b ” and “ C has d ” have no common constituent aside from the idea “has”, then it is obvious that, no matter which ideas we envisage as variable, these two propositions will never stand in a relation of deducibility, since the ideas which are placed in one of them are altogether independent of the ideas which will appear in the other.

It is clear that Bolzano here offers a counterexample to the claim that any given propositions stand in a relation of deducibility with respect to some selection of variables. It seems to us that there is no reasonable way to interpret what Bolzano says here as a proof of a general theorem about variable sharing (which he nowhere states). Rather, it seems to us that Rolf George was the first to state and prove the above result. One final remark: Bolzano offers a similar proof of a related result at WL, §375 [III.475–6].

Probability

We have seen that Bolzano made compatibility a condition of deducibility. Perhaps the most important motivation for this decision was the possibility of extending deducibility to a logic of probabilities, so that inductive and deductive logic could be united in a single system. Bolzano uses the expression ‘probability’ only for conditional probability, and sometimes uses the expression ‘comparative validity’ for it. He gives the following definition:

The conditional probability of a proposition M relative to the propositions $A, B, C, \dots$ (as always, with respect to variable constituents $i, j, k, \dots$) is the ratio of the number of cases obtained by variation where $A, B, C, \dots, M$ are all true to the number of cases where all of $A, B, C, \dots$ are true.¹

The probability of M is thus a fraction between 0 and 1. Now this fraction cannot be defined unless its denominator is non-zero, that is, unless at least one substitution makes all of $A, B, C, \dots$ true, or, in other words, unless $A, B, C, \dots$ are *compatible*.

¹ WL, §161 (our paraphrase).

As was the case with concept of the degree of validity of a proposition, Bolzano clearly recognizes that the set of appropriate substitutions needs to be constrained if the above definition is to give an accurate measure of probability. In addition, he discusses how to get a measure of probability in some cases where the class of allowable substitutions is infinite.¹

If the probability of M relative to $A, B, C, \dots$ and $i, j, k, \dots$ is equal to one, the number of substitutions that make $A, B, C, \dots$ as well as M true is equal to the number of substitutions that make $A, B, C, \dots$ true—but this occurs only if every substitution that makes all of $A, B, C, \dots$ true also makes M true, i.e., when M is *deducible* from $A, B, C, \dots$. On the other hand, if the probability of M is zero, then no substitution makes all of $A, B, C, \dots$ as well as M true, and M is therefore *incompatible* with $A, B, C, \dots$. The probability of M relative to $A, B, C, \dots$ thus has deducibility and incompatibility as extreme cases. This is an extraordinary achievement. Bolzano's approach yields the first logical definition of probability, and for the first time brings deductive and inductive logic together in a single theory. As Wittgenstein would later say: "the certainty of a logical inference is a limit case of probability."²

Excursus: Deducibility, Necessity and Apriority

It is important to note that, as with analyticity, deducibility as such carries no epistemological freight on Bolzano's conception. It is a relation between abstract objects, propositions in themselves, and may obtain whether or not we are aware of it. Again, too, the extreme generality of the definition ensures that there will be no temptation to claim that every case of consequence can be recognized *a priori*, or reflects a necessary connection. According to this definition, for example, the proposition

George H. W. Bush became president of the United States of America in 1989

counts among its consequences

George H. W. Bush's eldest son became President of the United States of America in 2001

because any substitution for X that makes

X became president of the United States of America in 1989

¹ For details, see **WL**, §161, nos 7 *et seq.*

² *Tractatus*, 5.15–5.152.

true also makes true

X's eldest son became President of the United States of America in 2001.

The above example, by the way, is by no means an isolated one as Bolzano explains in §110 of the *Theory of Science*. There he defines what he means by a *complete* idea of an object α :

[I]f the proposition “ α is A ” is true, then from this proposition either all, or only some of the attributes of α can be derived, where for this derivation nothing is used but truths of the form “ A has the attribute m ”, “ A has the attribute n ”, etc. If all attributes of α can thus be derived, then A is called a *complete* or *exhaustive* idea of its object; otherwise we say that A represents the object α only *incompletely*. Thus, I call the concept “a being that is omnipotent” a complete concept of God, since it is possible (perhaps not for man, but in itself) to derive from it all the attributes of God in the just indicated way, i.e., only by means of propositions of the form “an omnipotent being must also be omniscient, independent, etc.”. For the same reason I call the idea “father of Alexander the Great” a complete idea of King Philipp of Macedon, while the concept “King” and also the concept “King of Macedon” are examples of incomplete ideas of that man. For it is impossible to deduce from them anything approaching all the attributes of the object that they are to represent.¹

Subsequently, he points out that whenever an idea represents an object α exclusively, it is a complete idea of that object. His proof runs as follows:

If an idea represents only one object, then it is always an exhaustive idea of that object For if there are some attributes of the object α that cannot be derived from the idea A in the indicated way, then a second object β can be thought which has all attributes that can be derived from A but lacks those that are present in α but cannot be derived from A . Thus the object β too, would fall under the idea A , and A should not be called a singular idea.²

¹ WL, §110 [I.517–18].

² WL, §110 [I.518].

At first encounter, many readers might think that Bolzano is here endorsing a famous Leibnizian thesis:

Now it is certain that every true predication has some basis in the nature of things, and when a proposition is not an identity, that is to say, when the predicate is not expressly contained in the subject, it must be included in it virtually. [...] So the subject term must always include the predicate term in such a way that anyone who understands perfectly the concept of the subject will also know that the predicate pertains to it. This being premised, we can say it is the nature of an individual substance or complete being to have a concept so complete that it is sufficient to make us understand and deduce from it all the predicates of the subject to which the concept is attributed. An accident, on the other hand, is a being whose concept does not include everything that can be attributed to the subject to which the concept is attributed. Thus the quality of king which belonged to Alexander the Great, if we abstract it from its subject, is not determined enough to define an individual, for it does not include the other qualities of the same subject or everything which the concept of this prince includes. God, on the contrary, in seeing the individual notion or “haecceity” of Alexander, sees in it at the same time the basis and the reason for all the predicates which can truly be affirmed of him—even knowing *a priori* (and not by experience) what we can know only through history—whether he died a natural death or by poison.¹

Well, yes and no. Leibniz, as we can see from the above passage, held that in order to form an idea of an individual substance, one must form an idea from which all of the attributes of the substance can be deduced. In addition, however, he claimed that a perfect understanding of the idea would permit one to know every truth about its object.

Bolzano agreed with the first claim, but rejected the second. To form an idea of an individual like Alexander, he thought, we need not have in mind an idea which, if perfectly understood, would reveal every last detail of his biography. An idea of someone who possesses even one of the attributes peculiar to Alexander would suffice to represent him exclusively. Such an idea would, as we have seen, have to contain intuitions according to Bolzano.

¹ G. W. von Leibniz, *Discourse on Metaphysics*, 8; in L. Loemker, ed., *Philosophical Papers and Letters* (Dordrecht: Kluwer, 1989), pp. 307–8.

A contemporary of Alexander, for example, might have formed such an idea having the form [*this man*]. He could understand this idea with perfect clarity and distinctness and yet know very little indeed about its object. All the same, Bolzano endorses Leibniz's first claim, namely, that if the idea does represent Alexander exclusively, then every truth about Alexander is deducible from it. What can he possibly mean?

When Bolzano says that an attribute *b* is deducible from a given idea *A*, he means simply that there is a relation of deducibility between the propositions [*a* is *A*] and [*a* has *b*], where [*a*] is the only variable. Nothing in this definition guarantees that we, or anyone else (with the exception of God), can *know* that such a relation obtains in any particular case, still less that we can know this *a priori*.

This is why, in §155, no. 2 of the *Theory of Science*, Bolzano says that a relation of deducibility between propositions *A, B, C, D, ...* and *M, N, O, ...*

... is of particular importance, since *once we know that it obtains* it puts us in a position to infer the truth of *M, N, O, ...*, once we have recognized the truth of *A, B, C, D, ...*.¹

In marked contrast to what John Etchemendy calls "our ordinary concept of consequence," then, Bolzano's concept of deducibility contains no epistemological element. To say that something follows, on Etchemendy's view, seems to mean that one can know, indeed even *a priori*, that it does.² Not so for Bolzano.

To further illustrate this point, let us take a moment to explain some remarks that Bolzano made about intuitions. Recall that he had argued for the simplicity of intuitions as follows:

¹ WL, §155, no. 2 [II.113].

² *The Concept of Logical Consequence* (Cambridge: Harvard University Press, 1990), p. 83. Cf. p. 81: "The most important feature of logical consequence, as we ordinarily understand it, is a modal relation that holds between implying sentences and sentence implied. The premises of a logically valid argument cannot be true if the conclusion is false; such conclusions are said to 'follow necessarily' from their premises." Also (p. 89): "If you accept the premises of a valid argument, you must also accept the conclusion (to which we sometimes add 'on pain of irrationality')." This epistemic characteristic is sometimes thought to be more important than, and perhaps to underlie, our intuitions about the alethic modality involved in valid arguments. For example, some would claim, not implausibly, that it is only due to the *a priori* relation between the premises and conclusion of a valid argument that we judge the latter to follow necessarily from the former, and hence we judge the argument valid. On this view, a necessary consequence that could not be recognized as such *a priori* would never qualify as a logical consequence. And this certainly seems right."

But as certain as it is that ideas of the form “This, which now occurs in me” are singular ideas, just so is it certain that among these there are at least some that are completely *simple*. For if we suppress the thought of any additions like “which occurs in me just now”, “which I just now see, hear, or feel”, or “which I am now pointing at with my finger”, etc., the bare idea designated by the word “this” is certainly a completely simple idea. But the object that it represents remains throughout the same single one, whether we think the additions or not. For, if we consider them more closely, all these additions express no more than certain attributes that the single object that we just now represent possesses *precisely because it is this one and no other*; indeed, our idea does not become restricted to that single object only by means of these additions, but rather becomes redundant through them.¹

Exner was baffled by this claim:

[W]hen I say “this” (namely, an actual rose, which is red, plain, with yellow stamens, green, serrated, wilted leaves etc.), the question arises: are these characteristics all explicitly contained in that which is designated by “this” or not? No, for then that which is designated by “this” would be something composite; rather, they are contained in it in such a way that they can be derived from it as redundancies. But what would be the simple characteristic represented by “this” from which the rest could be derived? Can the redness of the petals be derived from their roundness or from the yellowness of the stamens, or which of these characteristics could ever be derived from another? . . . But according to your presentation, it seems that all characteristics arise from the one unknown characteristic designated by “this”, something that is incomprehensible to me.²

¹ **ML**, §6, no. 4 [**BBGA** 2A7, p. 59; **MM-EX**, p. 51] (emphasis added). In the *Theory of Science*, he makes it clear that the ‘because’ boils down to deducibility: “For the fact that this object has the attributes indicated in the additions, e.g., that it is a pleasant fragrance, already *follows from* the fact that it is precisely this object and not another which we represent” (§278 [III.22], emphasis added).

² Exner to Bolzano, 10 August 1833; [**BBGA** 3.4/1, p. 37; **MM-EX**, pp. 105–6]; cf **WL**, §278 [III.22]. Exner is not alone in his puzzlement. Cf. M. Textor, *Bolzanos Propositionalismus* (Berlin: De Gruyter, 1996), p. 93.

Exner seems, quite naturally, to suppose that when Bolzano says the properties of the object of an intuition can be derived or deduced from its idea (i.e., the intuition), he means that we can *see* that they follow, that the careful consideration of the idea will reveal all. But this is not what Bolzano meant. Rather, his point is that, supposing [this] to be an intuition which represents something that is red, every appropriate substitution for x which makes [x is this] true makes [x is red] true. The attributes follow, that is, in the sense of Bolzano's definition of deducibility.

Incidentally, the case of intuitions is somewhat different from that of other kinds of singular ideas. For an intuition cannot appear by itself as the predicate idea of any proposition; hence when, in §110 of the *Theory of Science*, Bolzano says that all intuitions are complete ideas of their objects, this cannot be understood to mean that if b is an attribute of the object of an intuition A , that every appropriate substitution for x which makes [x is A] true also makes [x has b] true, at least not if the 'is' in the former sentence is construed as the 'is' of predication. Rather, it seems to us, we have to do here with the 'is' of identity.¹ When Bolzano claims that the properties of the object of an intuition follow from its idea, he should be taken to mean that the following schema is valid for every b which represents a property of the object of the intuition [this]:

$$\frac{x = \text{this}}{\therefore x \text{ has } b}$$

Indeed, the simplest way to understand his view here would be to say that the fact that the object of an intuition has the properties it has follows from the fact that it is what it is; in other words:

[This has b]

follows from

[This = this]

where b represents any property of the object of the intuition and only the underlined occurrences of [this] are considered variable.

Having said all this, the epistemological side of deducibility or logical consequence was clearly also important to Bolzano. Even if, as his methodology required, epistemological considerations have no place in the *Theory*

¹ Another way to finesse this problem is suggested by **WL**, §225, no. 2: while an intuition ([this]) cannot occur as the predicate-idea, [the attribute of standing under the idea [this]] can.

of *Elements*, where propositions and ideas in themselves are the topic, they do arise later.

For Bolzano, recall, to say that a proposition M follows from propositions $A, B, C, \dots$ with respect to variables $i, j, \dots$ is just to say that every substitution for $i, j, \dots$ which transforms all of $A, B, C, \dots$ into truths also transforms M into a truth. To know that a relation of deducibility holds, then, is to know that a certain universal generalization is true.

But there are many kinds of universal generalizations, for example:

- All the members of city council are corrupt
- All reptiles have three-chambered hearts
- No body travels with a velocity greater than that of light
- Every substitution for [human], [vertebrate], and [mortal] that transforms the two propositions [All humans are vertebrates] and [All vertebrates are mortal] into truths also transforms [All humans are mortal] into a truth.

And different sorts of knowledge (which we may or may not be able to obtain) will be required to determine whether or not they are true.

In the case of relations such as deducibility, Bolzano tells us that we should consider the *form* of an inference or, what amounts to the same, we should focus on the *invariable* parts. He writes:

[I]f we must demonstrate that the propositions $M, N, O, \dots$ are *deducible* from $A, B, C, \dots$ with respect to the variable ideas $i, j, \dots$ (§155) we can by no means do so by showing that all of the infinitely many ideas which, when put in the places of $i, j, \dots$, make all of $A, B, C, \dots$ true, also make all of $M, N, O, \dots$ true. Rather, we must gather this from a consideration of the form of these propositions—either directly, or else through the consideration of other truths. The most important forms where a relation of deducibility can be recognized, either immediately or with the help of knowledge belonging to logic, have been discussed in the chapter dealing with inferences. Others, those where mathematical, historical, etc., knowledge is required to judge are not treated here.¹

¹ WL, §368 [III.451–2]; see §367 for similar remarks on analyticity.

Consider now the case of logical deducibility, using the last example given above:

All	humans	are	vertebrates
All	vertebrates	are	mortal
All	humans	are	mortal

Focusing on the form, we have:

All P are Q
All Q are R

All P are R.

Or, in symbols:

$\forall x(Px \rightarrow Qx)$
$\forall x(Qx \rightarrow Rx)$

$\forall x(Px \rightarrow Rx).$

And the obvious truth to consider here corresponds to the second-order formula:

$$\forall P \forall Q \forall R ((\forall x(Px \rightarrow Qx) \wedge \forall x(Qx \rightarrow Rx)) \rightarrow \forall x(Px \rightarrow Rx))$$

Clearly, though, this is what Bolzano would have called a purely conceptual truth, indeed, one that contains only *logical* concepts. Thus it is precisely of the sort which were recognized as candidates for *a priori* knowledge. In the case of the following form of inference, by contrast

X resigned from the US presidency in 1974

X had trouble finding competent plumbers

the corresponding universal generalization

Everyone who resigned from the US presidency in 1974 had
trouble finding competent plumbers

obviously contains a number of intuitional ideas, and hence is not the sort we should expect to know *a priori*.¹

¹ For further discussion of deducibility, apriority, and necessity, see P. Rusnock and M. Burke, "Etchemendy and Bolzano on logical consequence," *History and Philosophy of Logic* **31** (2010) 3–29.

Logical Concepts

We conclude our discussion of Bolzano's variation logic with a few remarks on logical concepts. We have seen that Bolzano distinguished special cases of analyticity and deducibility which might be called *logical* analyticity and *logical* consequence: a proposition is logically analytic if it is universally valid with respect to a form whose only invariable parts are logical ideas, and similarly in the case of consequence. In marked contrast to many later authors, Bolzano does not attempt to make a great deal of the distinction between logical and non-logical, or, in the language preferred by many of his contemporaries, between the formal and the non-formal. In our view, there are several reasons for this. First of all, we recall Bolzano's rejection of Kant's claim that logic was a completed science. Nothing could be farther from the truth in his opinion, and the number of new concepts appearing in Bolzano's *Theory of Science* was proof enough of the unfinished state of the discipline and its open-ended future. But if this is the case, attempts to precisely determine the limits of logic were almost certain to fail.¹ But if we are not in a position to say which concepts belong to logic and which do not, the distinction between logical and non-logical consequence and analyticity remains itself a rather imprecise notion.

We have seen that Bolzano's very general definition of consequence permitted him to declare many argument forms valid that others would have rejected as being at best enthymematic. In this general perspective, to be sure, there are some cases where the consequence relation seems pretty obviously logical, e.g.,

Some A are not B . Therefore, not all A are B

and some cases which seem obviously non-logical, such as

X was a twentieth-century US President who ordered the illegal bombing of Cambodia. So X once gave a sentimental speech about his dog.

But the boundary between the two can also seem quite blurred. Is there really such a great difference, for instance, between the following two inferences?

If A then B and if B then C . Therefore, if A then C .
 $a < b$ and $b < c$. Therefore, $a < c$.

¹ See **WL**, §116, 185, 254 for Bolzano's critical remarks on the attempts of other logicians to draw a sharp distinction between form and content.

Recall that Bolzano agreed to call logic a formal science only insofar as it dealt with entire classes of propositions, or arguments, etc. A form, in one sense of the word, is just a class of propositions (or arguments, etc.); more precisely, it is a linguistic expression containing variables that is associated with a class of propositions. Thus for example, the form ‘A is honest’ determines the class

{Joe is honest, Sarah is honest, ...}

The formal aspect of, say, a proposition, is just the part that all the members of the class have in common. But every proposition has several forms, and hence the formal aspect of a proposition depends upon the way you look at it: there is no absolute separation of formal and non-formal elements. The appellation “logical”, for its part, does not seem for Bolzano to mark out a region of thought with any special epistemological or metaphysical status (apart from that which logic shares with all purely conceptual sciences). Indeed, Bolzano thought that the boundaries between sciences should be settled in large measure by *pragmatic* considerations: various concepts or propositions are treated in one and the same science in part because it is *useful* or *beneficial* to proceed in this way. It is advisable to have a science of logic, a science dealing with the organization and presentation of scientific information, and it is far from obvious that all the concepts in such a science will have the same, special epistemological status. To declare a concept logical is, on this view, to say that it is a concept that logicians would do well to discuss. His primary preoccupation does not seem to have been—as some earlier commentators have thought it must and should have been—to arrive at the concept of *logical* consequence or *logical* analyticity. Rather, these are special cases of more general and more interesting relations, singled out because in a treatise on logic, it is reasonable to expect that only inferences grounded on logical concepts are dealt with:

[A]ccording to the very wide sense in which I have taken the word deducibility (§155) the validity or invalidity of some deductions can be assessed only if we have knowledge of matters outside logic. Thus from the proposition “this is a triangle” we may deduce the proposition “this is a figure the sum of whose angles equals two right angles” (with respect to the idea “this”), and from the proposition “Caius is a man”, we can deduce the proposition “Caius has an immortal soul” with respect to the idea “Caius”). For whenever we replace the indicated idea by some other idea, the conclusions become true whenever the premises

are true. But to realize this, we must know two truths, namely that the sum of the angles in any triangle equals two right angles, and that the souls of all men are immortal. Since these are truths which are not at all concerned with logical objects, i.e., with the nature of concepts and propositions, or rules according to which we must proceed in scientific exposition, nobody will demand that logic should teach deductions of that sort. Hence, what can be expected in this place is only a description of those modes of deduction whose correctness can be shown from logical concepts alone, or, what comes to the same thing, which can be expressed in the forms of truths, in which nothing is mentioned except concepts, propositions, and other logical objects.¹

This passage reveals relatively little. Logic, apparently, deals with the nature of propositions and concepts (or more generally ideas). What, exactly, are the “other logical objects”? Do linguistic entities such as the propositional forms, for example, count as logical objects given that Bolzano presents the theory of inferences in terms of them? Even if, as seems to be the case, Bolzano did not think the boundary between logic and other disciplines was of deep metaphysical or epistemological significance, it is still interesting to make a provisional inventory of his logical concepts.

I. Auxiliary concepts, mereological and metalogical concepts

1. The concept of an *object*, or *something* (*Gegenstand*, *Etwas*, *Etwas überhaupt*), of an *attribute*, which covers both *properties* (attributes of single objects) and *relations* (attributes of collections). For Bolzano, everything is an object, including the properties and the states of things.
2. Mereological and set theoretical concepts: The concept of a *collection* (*Inbegriff*), of *whole* and *part*, of a *multitude* or *set* [*Menge*], an *extension*, of a *sum* and of *content*.
3. Fundamental metalogical concepts: *proposition in itself*, *idea in itself*, *concept*, *intuition*, as well as concepts of constituents of propositions such as: *subject-idea*, *predicate idea*, *copula*, the concept of a *concretum* (an idea of the form [something, which has the attribute *b*] and an *abstractum* (an idea of an attribute).
4. Fundamental semantic concepts: *objectuality* (*Gegenständlichkeit*) and *objectlessness*, *truth*, and *falsity*.

¹ WL, §223.

5. Concepts expressing variation: *place of variation, variable idea, substitution, making true/false*.
6. Concepts of logical relations: *identity and diversity, inclusion, subordination, equivalence, incompatibility, exclusion, contrariety, contradiction, deducibility*, and related concepts.

II. Logical concepts strictly speaking

1. Linking concepts: the copula *has* (to possess an attribute), and concepts that link the parts of certain ideas: *who, which, and lack of* (predicate negation).
2. Concepts of operations on ideas: *not-, and, or*.
3. Quantifiers (which Bolzano does not always translate into his canonical language): *all, each, whichever, several, a few, none, one, two, etc.*
4. Propositional connectives: *either ... or, if ... then, and, not* (proposition negation).

It goes without saying that this list is incomplete and, as Bolzano says, not immune from controversy. Among other things, we have included only concepts from elementary logic: in order to fill it out, it would be necessary to add modal operators, probabilistic concepts, and many others.

8. GROUND-CONSEQUENCE AND THE OBJECTIVE CONNECTION BETWEEN TRUTHS

Bolzano had a rich variety of concepts of consequence at his disposal—ordinary deducibility, logical deducibility, exact deducibility, exact logical deducibility, etc.—enough, one might have thought, to satisfy the requirements of even the most fastidious methodologist. Yet, when he considered what he called the objective connections between truths, he thought that more remained to be said. In particular, he believed that there was a relation of objective dependence between truths that differed from all of the varieties of deducibility he had distinguished, and that it was precisely this relation that many philosophers had been aiming at in their studies of the order of propositions in a science.

He gave the name of *ground-consequence* [*Abfolge*] to the relation of immediate objective dependence of one truth on one or several others.¹ He

¹ See **WL**, §162, §§198–222; also, **ML**, §§13–14 [**BBGA** 2A.7, pp. 81–8; **MM-EX**, pp. 69–75].

seems to have thought that the ground-consequence relation was in some respects like deducibility, in some like causality, and in others like “providing an explanation for”. Not surprisingly, he did not succeed in uniting these quite heterogeneous intentions into a single coherent theory. Bolzano tells us that a certain uncertainty plagues everything he says on the subject, and warns us not to expect anything by way of a complete account, his goal being to draw the reader’s attention to an important unsolved problem.¹ We will give a sketch of his remarks.²

The order observed in axiomatic presentations of the most advanced sciences, e.g., mathematics, is an order of *proof*, and so it might be thought that the relation of ground-consequence could be defined in terms of the relation of deducibility. However, Bolzano thought it clear that deducibility alone would not do the job, not even the narrower notion of exact deducibility.³ For one thing, he seems to incline towards the view that there are instances of the ground-consequence relation that are not also cases of deducibility.⁴ Moreover, there are cases of mutually deducible, or equivalent, propositions where it seemed clear to Bolzano that the ground-consequence relation could only go one way. Consider, for example, the following two propositions:

- (1) [The barometer reads higher today than yesterday.]
- (2) [The air pressure is higher today than yesterday.]

Supposing the barometer to be functioning properly, these propositions are mutually deducible when [today] and [yesterday] are considered variable. What is more, knowing (1) allows us to recognize the truth of (2); (1) is what Bolzano calls a subjective ground or “ground of knowledge” for (2). On the other hand, he thinks it absurd to say that (1) is an *objective* ground for (2); rather, (2) should be thought of as a (partial) *objective* ground for (1):

[W]e will never believe that the truth that the barometer stands lower today than yesterday is the *objective ground* of the truth

¹ WL, §195.

² See also G. Buhl, *Ableitbarkeit und Abfolge in der Wissenschaftslehre Bolzanos*, *Kantstudien* Ergänzungshefte **83** (1961); P. Mancosu, “Bolzano and Cournot on mathematical explanation,” *Revue d’histoire des sciences* **52** (1999) 429–55; S. Roski, *Bolzano’s Conception of Grounding* (Frankfurt am Main: Klostermann, 2017); and Sebestik, *Logique et mathématique chez Bernard Bolzano*, Part 2, Chapter 4.

³ WL, §200.

⁴ See, e.g., WL, §200 [II.348].

that the atmospheric pressure is lower today than yesterday. Rather, it is obvious that between these two truths it is the converse relation that holds; the latter truth is one of the partial grounds from whose connection the former is produced as a consequence: *because* the air pressure has decreased, the mercury has sunk.¹

The causal language is no accident in this case. Indeed, Bolzano thinks that causal statements are at bottom just statements of ground/consequence relations:

“*X* is the cause of *Y*” actually means “the truth that *X* exists is related to the truth that *Y* exists as ground (partial ground) to consequence (partial consequence).”²

But the relation of ground-consequence is not limited to, nor is it reducible to, cause–effect relations, for Bolzano thinks that it also applies to propositions dealing with things that have no causal relations, e.g., propositions in themselves or mathematical objects.³ He claims, for example, that although the following two propositions are mutually deducible relative to the variables [*A*] and [*B*], the former should be considered the consequence of the latter but not vice versa (we shall see why he thought this in a moment).

- (1) [A pair of circles in the same plane, one with centre *A* and radius *AB*, the other with centre *B* and radius *AB*, must intersect.]
- (2) [*A* and *B* being distinct points, there exists a third point *C* at distance *AB* from both *A* and *B*.]

In general, while deducibility may be mutual, Bolzano thinks that ground-consequence is anti-symmetric. Other structural features also separate deducibility from the ground-consequence relation: any proposition that is not universally invalid with respect to given variable ideas is deducible from itself with respect to the same ideas, but no proposition is its own ground. Finally, deducibility is transitive (at least with respect to the same variables), while ground-consequence is anti-transitive.⁴

We have seen that in some cases causal relations are associated with the ground-consequence relation. But this cannot be the case in, e.g., mathematics. How is the relation determined there? Bolzano frankly admits that he has

¹ **ML**, §13 [**BBGA** 2A.7, p. 82; **MM-EX**, pp. 69–70].

² **WL**, §168.

³ **WL**, §201.

⁴ **WL**, §§209, 204, 213.

no completed theory. However, he does have several suggestions. In §221 of the *Theory of Science*, he points to two features, simplicity and generality, which place some conditions on ground-consequence relations in purely conceptual sciences such as mathematics. A proposition *A* is called *simpler* than *B* if the content (p. 217 above) of *A* is a proper subset of the content of *B*. *A* is called *more general* “when either its subject or its predicate idea, or both, are of greater extension.” He sets out the following rules:

The simpler truth cannot be the consequence of the more complex.¹

Among truths of equal complexity, the more general truth cannot be the consequence of the more specific.

To illustrate the use of the first of these criteria, let us return to the two propositions mentioned above:

- (1) [A pair of circles in the same plane, one with centre *A* and radius *AB*, the other with centre *B* and radius *AB*, must intersect.]
- (2) [A and *B* being distinct points, there exists a third point *C* at distance *AB* from both *A* and *B*.]

(1) should be considered the consequence of (2), Bolzano argues, but not vice versa.² Having settled on the view that geometrical objects should be defined as structured point-sets, Bolzano saw that to say that two circles intersect is just to say that two point-sets have an element in common. But to say that the set of points at distance *AB* from *A* and the set of points at distance *AB* from *B* have a common member we first have to know that there is a point such as described in (2), and we can know this without even considering the sets in question. (2) is thus a *simpler* proposition than (1), since it does not involve the notion of set, and thus cannot be a consequence of (1).

To illustrate the second criterion, we refer to Bolzano’s observation that the theorems of the general theory of quantities (including arithmetic, algebra, and analysis) cannot be proved by appealing to *geometrical* truths.³ A proof of a theorem of analysis based on geometrical considerations could not

¹ **WL**, §221, no. 2. Bolzano restricts this claim to purely conceptual propositions, such as those of mathematics.

² **ML**, §13 [**BBGA** 2A.7, p. 82; **MM-EX**, p. 70]; cf. **WL**, §525; also “Versuch einer objectiven Begründung der Lehre von der Zusammensetzung der Kräfte,” Prague, 1843, §8 [**BBGA** 1.18, p. 22]; “Anti-Euklid,” ed. K. Večerka, *Sborník pro děj. přír. věd* **11** (1967) 203–16, p. 211.

³ See below, Chapter 9, p. 521 *et seq.*

possibly reflect the objective relations of dependence, since spatial quantities are just a special kind of quantities in general. To attempt to prove a theorem of analysis based on considerations from the less general science of geometry would thus be like trying to prove that no one in New York is over seven feet tall based on the observation that no one in a certain apartment building in that city is.

Finally, Bolzano offers what he considers a sufficient condition for truths $A, B, C, \dots$ to be the grounds of another truth M .¹ M is a consequence of $A, B, C, \dots$, he states, if M is *exactly deducible* from $A, B, C, \dots$ relative to variable parts $i, j, k, \dots$, and if $A, B, C, \dots$ are the *simplest* of all the propositions equivalent to $A, B, C, \dots$ relative to $i, j, k, \dots$. We can see the general thrust of Bolzano's inquiry: exact deducibility involves deducibility as well as the independence and indispensability of the premises. The criterion of simplicity provides the means to choose between rival premises meeting the previous criteria.

As Bolzano himself conceded, there are large gaps in this account of ground-consequence. One might even wonder if Bolzano's problem is sufficiently well-defined to admit of a solution. He ends his discussion with the following remark:

I occasionally doubt whether the concept of ground and consequence, which I have above claimed to be simple, is not complex after all; it may turn out to be none other than the concept of an ordering of truths which allows us to deduce from the smallest number of simple premises the largest possible number of the remaining truths as consequences.²

9. SCIENCES AND THEIR TREATISES: THE THEORY OF SCIENCE PROPER

The fourth and final volume of the *Theory of Science* is devoted to what Bolzano calls the theory of science properly speaking. Given all the abstract speculation that precedes this part, one can be forgiven for expecting the same sort of approach to be taken in this part. But we do not find anything of the sort. Instead, human needs take centre stage, a science being defined as

a collection of all the truths of a certain kind, which are of such a nature that the part of these truths that are noteworthy and known

¹ WL, §221, no. 7.

² WL, §221 [II.388].

to us is worth writing down in a special book, combined, if necessary, with other truths necessary for the understanding or proof of the former, in a way that makes them as comprehensible and convincing as possible.¹

The measure of what is worthwhile, moreover, is provided by the highest moral law, and the highest principle of the entire theory of science is accordingly an ethical one:

In dividing the entirety of truth into individual sciences and in presenting these sciences in special treatises, everything must be done in the way required by the laws of morality, thus in such a way that the greatest possible sum of good (the greatest possible promotion of the general well-being) is thereby produced.²

The boundaries of sciences, for example, are determined by practical considerations. We must always ask whether it would be useful to combine such and such truths into a single science. This usually amounts to asking whether there are circumstances in which a sufficient number of people would find it useful to learn precisely this collection of truths (or at least the part that is already known). We are clearly a long way away from “sciences in themselves” delimited on a purely objective basis. Instead, we find that Bolzano’s conception of the sciences rejects a variety of traditional claims: sciences are not, in general, occupied with a single genus of objects, they need not proceed from a single source of knowledge nor be based upon a single principle. Moreover, sciences often overlap, and enter into complicated relations of dependence, including cases of mutual dependence.³ There is no elegant, objective structure constituted by the entirety of sciences.

Within a presentation of a science (a treatise or textbook), moreover, propositions need not be ordered to reflect relations of ground and consequence or even deducibility. With different sciences, and different classes of readers for whom a treatise is intended, different approaches will be appropriate. Proofs may follow the objective order of ground-consequence relations, but they need not. It might even be appropriate to use false premises in some of our deductions, for example, in proofs *ex concessis*, where one shows that even on the (false) presuppositions of one’s readers, a given proposition still follows (though he adds that this should never be done in a way that suggests that the author himself accepts the false premises).⁴

¹ WL, §393 [IV.6].

² WL, §395 [IV.26].

³ WL, §§415–20.

⁴ WL, §534.

Principles or *basic propositions* [Grundsätze], moreover, are distinguished from *basic truths* [Grundwahrheiten]. While the latter, as we have seen, are defined objectively, in terms of the ground-consequence relation, the former are defined in terms of the role they play in a treatise, and thus once again ultimately in terms of human needs. In particular, some principles are merely subjective, used only in order to convince, rather than to make us aware of objective connections between the truths in the science.¹

Similar remarks apply to proofs, classifications, the divisions of a treatise and other features besides. Consistently, the approach is pragmatic, and the question always before Bolzano's mind is: "Would the adoption of this feature in a treatise help in organizing and communicating useful knowledge?"

Bolzano recognized that he was out of step with his contemporaries on these points just as he was when he took an abstract and objective approach in the *Theory of Elements* while most of them developed logic on a subjective or psychological basis:

My criticism of the now customary form of presentation of logic that ideas, propositions and truths are never considered from the objective point of view, is aimed only at the part of logic usually entitled the *Theory of Elements*. Concerning the so-called *Theory of Method* I would advance the opposite reproach, namely, that one abstracts too much, in that one presents only the laws of thought that apply to all beings, and says nothing of the ones that concern only us humans. In the *Theory of Method*, one should present the rules that tell us how to produce a science, or rather, a treatise of a science. A treatise is supposed to be a book in which the truths belonging to a given science are written in a way that makes them as comprehensible and convincing as possible, not for *every* thinking being (e.g., not for angels), but rather for us humans. Thus in order to give a complete specification of the rules which must be followed here, one must not pay attention only to the conditions governing thought and knowledge in all beings, but also to those valid for *human beings*. The *Theory of Method* should not therefore in my view be considered part of *pure* logic, but rather united with *applied* or empirical logic, and its doctrines should be presented not before, but after those of pure logic as their application.²

¹ WL, §§483–91.

² WL, §16 [I.66–7].

In certain cases, Bolzano thought, pragmatic considerations justify presenting a science in a way that shows as much of its objective structure as possible. Treatises organized in such a way are thus also envisaged as a special case in Bolzano's general theory of science. In such presentations, we would do our best to identify all simple concepts occurring in the science, to define all others in terms of these, to identify not just subjective principles but the most objectively fundamental propositions of the science (basic truths even, in some cases), and to strive to make our proofs indicate the objective connections (relations of ground-consequence) that hold between the truths we are concerned with. In this special case, Bolzano comes close to the ideal that Betti, De Jong, and others have called the classical model of science.¹

Bolzano thought this approach particularly appropriate in the case of pure mathematics, and spent many decades working on developing mathematics in precisely this way. As we intend to speak later of Bolzano's mathematical work, however, we will postpone until then our discussion of the features of this special kind of scientific presentation.

10. CONCLUSION

An integral part of Bolzano's *Theory of Science* is historical—"constant attention to earlier contributors to the field", as he put it in the subtitle of the work. The promise is kept, perhaps a little too well for Bolzano's modern readers, who, if they wish to read the work in its entirety, must wade through lengthy discussions of Bolzano's predecessors and contemporaries. Though many of these do not contribute much to our understanding of logic, they do serve to highlight the degree to which Bolzano stood apart from his contemporaries.

Bolzano could not have been more out of step with the main philosophical currents of his times (particularly in Germany)—often called the Age of German, or Post-Kantian, Idealism. Indeed, to judge from the majority of histories, he seems to have been born a hundred years too soon. These tell us that analytical philosophy was a relatively late arrival on the scene, appearing in Jena and Cambridge after Hegelianism and related movements had begun to run out of steam. Bolzano's works put the lie to these accounts. In them one finds not only many key points of the program of classical analytic philosophy, but also a great many of the significant details. With him, Central European philosophy started on the analysis that would dominate world phil-

¹ See, e.g., *Synthese*, Vol. 174, no. 2, May 2010: *The Classical Model of Science I: A Millennia-Old Model of Scientific Rationality*.

osophy in the second half of the twentieth century. Because this origin was obscured and because Frege's acquaintance with philosophical tradition was rather weak and mostly second-hand, analytical philosophers, trained mainly in logic and with insufficient knowledge of the past, forgot that they were heirs to the great tradition of Aristotle and Leibniz and were often forced to re-invent solutions proposed by their predecessors. As Husserl realized, Bolzano's philosophy is essential for modern analytical philosophy, because it links it with the great philosophers of the past, opening up at the same time wide perspectives towards the future.

In his book on the origins of analytical philosophy, Michael Dummett deplores the "prevalent modern habit of speaking of analytical philosophy as 'Anglo-American'. [...] This terminology distorts the historical context in which analytical philosophy came to birth, in the light of which it would better be called 'Anglo-Austrian'..."¹ It also obscures the fact that modern analytic philosophy and phenomenology have a common root. All of this would have been obvious, he continues, had it not been for the political collapse of central Europe in the twentieth century.

When he has been mentioned at all in histories of nineteenth-century philosophy, Bolzano has usually seemed out of place. Copleston's *History of Philosophy* is instructive here. For the most part, his account of the nineteenth century follows the usual pattern. It is the century of Kant's successors, Fichte, Schelling, and Hegel occupying pride of place. Attentive to the facts, though, he cannot avoid saying something about Bolzano, even though he cannot really fit him into his narrative. "Chronological reasons," he writes, "justify the inclusion in this chapter of some brief reference to Bernard

¹ M. Dummett, *Origins of Analytical Philosophy* (London: Duckworth, 1993), pp. 1–2. Dummett continues: "In central Europe [...] there were throughout the nineteenth century a great many diverse currents in philosophy, which did not, however, flow along isolated channels, but collided with each other because of the communication between representatives of the different trends in the universities. More than one of these currents contributed, in the twentieth century, to the formation of analytical philosophy, which, before Hitler came to power, was to be viewed as more a central European than a British phenomenon. The shifting of the scientific and philosophical centre of gravity across the Atlantic [...] was principally a long-term effect of political events, that is, of the Nazi regime which drove so many to take refuge in America: the process is now being completed by those many contemporary European governments that have set themselves to inflict the maximum damage on their countries' university systems. That, of course, does not make it any the less real; but it is a grave mistake to project present realities back into a past in which they were as yet unimaginable."

Bolzano, even if his rediscovery as a forerunner in certain respects of modern logical developments tends to make one think of him as a more recent writer than he actually was.”¹ Surely, though, the problem here is not with Bolzano, but rather with the historians’ expectations. Because of his detailed knowledge of both mathematics and philosophy (a rare combination of specialities even today), Bolzano was among the first to see some of the characteristic problems of contemporary philosophy. On account of his intelligence, independence, and doggedness, he was also the first to achieve decisive results—results which, far from being of merely historical interest, still have in many cases things of value to teach us today. A philosopher of Bolzano’s stature is not a minor anomaly that can be safely ignored. If he doesn’t fit into the framework of the standard historiography, then so much the worse for it.

¹ *A History of Philosophy* (London: Burns and Oates, 1965), Vol. 7, p. 256.