Quotes from Bolzano's *Wissenschaftslehre*

(with occasional annoying notes by Bob Brandom)

From Rolf George abridged edition:

From Editors Intro:

Bolzano held, though he did not claim to have a proof, that all propositions have the same basic structure, namely 'A has *b':* they all assert that a subject has a certain character. (Bolzano generally uses capitals for *denotative*, and lower case letters for *attributive* expressions.) He expands not a little ingenuity on fitting various expressions into this pattern: 'some *A* are *b'* becomes 'The idea of an *A* which has *b* has reference' ; *'p* v *q'* becomes 'The idea of a true proposition among *p* and *q* has reference'. Without doubt, Bolzano's insistence on a common form for all propositions was detrimental to the development of several aspects of the system. W. and M. Kneale have pointed out that the development of argument patterns, the calculational aspect of logic, was not notably advanced by Bolzano, and they justly cite his rigidity in matters of logical form as one of the reasons.* In particular, since the logical connectives are absorbed into the predicate of standard form propositions, Bolzano "has made it has made it necessary to express notions which are usually regarded as formal (e.g. those of negation and particularity), by signs which enter his reductive formulae in the same way as signs which would ordinarily be said to express material notions (e.g. that of wisdom)." [xxxiii]

the form of a proposition is a class of propositions that may be generated from it by replacing a certain constituent idea by other ideas. [xxxiv]

Bolzano's theory of deducibility forms the core of the *Theory of Elements.* He was the first to give a formal definition of the notion of consequence. It is akin to that given a century later by Tarski. Bolzano defines deducibility as follows: "Propositions *M, N, O,. ..* are deducible from propositions *A, B, C, D,. . .* if every class of ideas whose substitution for *i,j*,... makes all of *A, B, C, D,. . .* true also makes all of *M, N,* O,.. . true." Tarski's definition of consequence (actually a preliminary version) is: "The sentence *X follows logically* from the sentences of class *K* if, and only if, every model of the class *K* is also a model of the sentence X."§ A model for class *K* of sentences is any set of objects which satisfies the propositional functions generated from *K* by replacing all extralogical constants of *K* by variables.

the main difference lies in the absence of the notion of a function in Bolzano and the fact that he does not draw a sharp dividing line between logical and extralogical parts of propositions. Thus, when he speaks of variable ideas, he may [BB: ?] mean any parts or particles whatever of the propositions in question. [xxxv]

Another major difference is that in Bolzano **deducibility is always relative to a set of variable (in his sense) ideas**.
Thus, the concept of deducibility becomes much wider in Bolzano than it is in Tarski.
Consider the argument 'Socrates was a man, therefore Socrates was mortal'. It is valid in Bolzano's sense relative to the idea 'Socrates' since every substitution on 'Socrates' which makes the premises true also makes the conclusion true. But we can know this only if we first know that all men are mortal. Hence to assess the validity of an argument it will generally be necessary to have a good deal of extralogical knowledge.
Tarski, by contrast, asks us to turn *all* extralogical constants into variables, hence knowledge of mortality or humankind will not be required to assess validity. On the other hand, since Bolzano does not distinguish logical and extralogical parts of propositions, he would presumably also permit substitutions on the former. This would make some arguments that are valid in Tarski's sense Bolzano-invalid relative to their logical particles; e.g. *modus ponens* is invalid with respect to 'if-then'. [xxxv]

[B's denial of *ex contradictione quodlibet*: explosion.]
A further important distinction between the two concepts of deducibility derives from the fact that in both cases the definiens is a universal affirmative proposition, i.e. a proposition of the form 'All *S* are *P'.* Most contemporary logicians, certainly Tarski, regard such propositions as true when there is no *S.* Bolzano, on the other hand, regarded them as false under these circumstances. It follows that if there is no model for the class *K* in Tarski's definition, then the definiens is true, hence *X* follows from the class *K.* This simply means that **if the set of premises is inconsistent, anything follows from it**. By contrast, Bolzano's interpretation of universal affirmative propositions makes his definiens false if there is no class of ideas whose substitution for *i, j,* . . . makes all of *A, B, C, D,* . . . true. **Hence for Bolzano nothing follows from an inconsistent premise set**. It is important to notice that Bolzano did not hold the truth of universal affirmative propositions to be *undetermined* if their subject terms did not have a referent. He took them to be *false* under this condition. This was a considered position which is sustained throughout W.L. For instance, in order to maintain it, he abandons the view that *A* and *O* propositions are contradictory. [xxxvi]

Still, it is remarkable that he clearly distinguished between objective measures of the probability of a proposition, and subjective strength of conviction.

After the concept of probability, Bolzano introduces another relation which he calls *Abfolge* (ground-consequence). This relation is said to hold between truths, but no definition of *Abfolge* is attempted; the relation is introduced only by way of examples.
From Bolzano's discussion we gather that the formal differences between deducibility and ground-consequence are that the former is transitive, non-symmetrical and simply reflexive, while the latter is intransitive, asymmetrical, and irreflexive. [xxxvii]

Bolzano's insistence that truths are ordered in themselves according to the relation of ground and consequence, and that this ordering is independent of our order of recognition has tended to re-enforce the Platonizing interpretation of Bolzano. [xxxviii]
[Cf. Ulf on "certainty." Both are picking up Aristotle.]

Bolzano next considers propositions and ideas as they are manifested in the mind…
Bolzano assumes, perhaps naively, that a judgment is the manifest presence of a proposition in itself in the mind. The proposition is said to be the matter, or stuff, of the judgment. Since propositions are composed of ideas in themselves, judgments are said to consist of subjective ideas which pass through the mind, one after the other, though the judgment is not merely the presence of a series of ideas, but also contains an element of affirmation or acceptance. Being actual affections of the mind, judgments and subjective ideas are actual, occur in time and have duration. Unlike objective ideas, subjective ones can be clear or obscure, distinct or confused, and more or less vivid; judgments can be formed with more or less confidence. [xxxix]

Terms of propositions *refer* to their objects. They do not need to resemble them.

The important problem is no longer whether our ideas match or resemble their objects, but whether our judgments are true, and how we come to make true judgments. The task of the theory of knowledge becomes to explain how true and false judgments arise in the mind; since Bolzano took a judgment to be composed not of sensations and similar entities (though it can of course be about sensations), but of mental terms, i.e. the mental counterparts of logical entities, he took the clues for his epistemology more from logic than from psychology. [xl]

<u>Intuitions:</u>
Bolzano's view is that intuitions are ideas that are both simple and singular. By this he means that intuitions do not consist of further ideas, and that they have precisely one referent. [xl-xli]

An intuition is said to be commonly designated by the word 'this', and to have as its object a change that "just now takes place is us". This change is also said to be the "immediate hence unanalysable cause" of the idea. [xli]

Concerning the origin of judgments, Bolzano distinguishes immediate and mediated judgments. He makes the claim that all immediate judgments are infallible, but, characteristically, attempts to show that this is a purely conceptual truth which can be established without even citing an example of an immediate judgment. [xli]
What is remarkable about this argument is that Bolzano does not argue from the phenomenal character of some judgments to their immediacy and infallibility, but argues for the infallibility of some forms of judgment from the fact that we know anything at all. [xlii]

Principles of selection. In Bolzano's letter to Romang, mentioned earlier, he gives advice concerning the reading of W.L. After advising Romang that he can lay aside the fourth volume, he continues "You can omit the entire book titled Heuretic in the third volume, and the lengthy part on arguments in the second book. It will perhaps be sufficient if you read in the first volume §§ 19, 25, and 26 (on the concept of propositions and truths in themselves) 48, 49, 50 (about ideas in themselves) 55, 56, 57, 58, 63, 64, 66, 67, 68, 70, 72, 73, 79 (time and space), 85 (sequence) 87 (where you find the concept of infinity which I put in the place of the self-contradictory Hegelian absolute). From the second volume §§ 125, 127, 133, 137, 148 (analytic and synthetic propositions)**, 154-58 (propositions with variable parts, where I define the relation of deducibility: whenever *A, B, C,*. . . are true, then *M* is also true),** 170 (propositions of the form 'a certain *a* has *b'),* 179 (propositions with if-then), 182 (the important concepts of necessity and possibility), 183 (time determinations), 197 (analytic and synthetic truths), 198 (concept of ground and consequence between truths), 201, 202, 214, 221 (basic truths). From the third volume all you need to read is what I say about the concepts of clarity and distinctness (280, 281), and then you may lay the entire logic aside, provided only that you look up one or the other item as the occasion requires; the index in the fourth volume will simplify this considerably." [xlv]
[Might adopt the policy of only reading these core sections. Sebestic basically agrees as to what the core sections are:]
Sebestic review: Before studying Casari's book, I recommend that readers first have a look at Edgar Morscher's 'Bolzano's Wissenschafstlehre' in *Morscher 2007* and then study Bolzano's text, especially vol. I, §§54–108, vol. II, §§132–68 and §§195–223 (§220 contains derivation trees), the heart of the matter being §§154–68.

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Quite different from the hitherto existing opinion was not only the *definition* but also the *concept* of our science which Hegel has brought forth, saying that it "is to be taken as the system of pure reason, as the realm of pure thought, in general, as the pure science which presupposes the liberation of the antithesis of consciousness, and **which contains the thought in so far as it is identical with the thing and the thing inasmuch as it is identical with the pure thought**." [10]

It is the task of logic to give rules which apply simultaneously to several truths or, what amounts to the same, to a whole class of truths.

For this reason, the theorems (though perhaps not the examples) of logic never concern a particular, fully determinate proposition, i.e. a proposition in which subject, copula and predicate are all given. Rather, theorems concern a whole class of propositions at once, i.e. propositions some of whose parts are determined, while the remainder is undetermined. Thus the proposition "some people have white skin" occurs in logic at best as an example, and not as the subject of a theorem, while a class of propositions, such as the class determined by the expression 'Some *A* are *B'* may well be the subject of a theorem. If these classes of propositions are to be called general *forms* of propositions, then it is permissible to say that logic is concerned with forms rather than with individual propositions. (Actually, only the written or oral expression 'Some *A* are *B',* and not the class itself, should be called a form.)

Furthermore, **if we want to call what is left indeterminate in such a class of propositions the *content* of the propositions in the class, such as the *A* and *B* in the above example, then we may say that logic is concerned merely with the form, and not the content, of propositions. (Actually, the parts of these propositions which are determinate have, in certain respects, the same claim to be called content as do the parts left indeterminate.)**
I should not object to calling logic a formal science, if what is intended is this feature. [13-14]

One must distinguish between the *objects* of a science (what it deals with or treats) and the *content* of a science (its theories). For example, the object of geometry is space, the contents of geometry are propositions about space. [14]

Logic is to teach us rules by which our knowledge can be organized into a scientific whole. [16]

…if an exposition of logical theory starts out by defining "ideas, propositions and truths as phenomena in the mind of a thinking being, it cannot possibly arrive at a true notion of the connections that obtain between truths in themselves. It will continue to confuse this connection with relations between experiences." [17]
[Contra psychologism.]
the so-called doctrine of elements does not generally treat representations, propositions and truths with sufficient abstraction, the opposite objection can usually be made to the chapters on methodology. [17]

I want to define first what I mean by a *spoken* proposition or a *proposition which is expressed in words.*
With this name I wish to designate any *speech act*, if through it anything is asserted or expressed; that is to say, whenever it is one of the two, either true or false in the usual sense of these words ; or, as one can also say, if it is either correct or incorrect. (Such speech acts will usually consist of several words, but occasionally only of one.) [20]

In other words, *by proposition in itself* I mean any assertion that something is or is not the case, regardless whether or not somebody has put it into words, and regardless even whether or not it has been thought. In the following example the word 'proposition' [20-21]

Just as one should not think of a proposition as something which is proposed by somebody, he should not confound it with the *idea* which is present in the consciousness of a thinking being, nor with a *belief* or a *judgment.* [21]
For this reason one must not ascribe being (existence or reality) to propositions in themselves. Only the mental or asserted proposition, i.e. the thought of a proposition, likewise the judgment which contains a given proposition, has existence in the mind of the being that thinks the thought or makes the judgment. [21]
[This is related to but not identical with the force/content distinction.]

a proposition in itself can be about thoughts and judgments, although it is itself neither a thought nor a judgment. [21]

[p. 22-3 on B's response to liar paradox.]
The word 'proposition' was preferred to 'judgment' *[Urteil],* 'statement' *[Aussage]* and 'Assertion' [*Behauptung*] since the others carry stronger overtones of agency. [24]

§25:
I shall mean by a truth in itself any proposition which states something as it is, where I leave it undetermined whether or not this proposition has in fact been thought or spoken by anybody. In either case I shall give the name of a truth in itself to the proposition, whenever that which it asserts is as it asserts it. In other words, I shall give it the name of a truth in itself whenever the object with which it deals really has the properties that it ascribes to it. [32]
They do not have actual existence, i.e. they are not something that exists in some location, or at a some time, or as some other kind of real thing. Recognized or thought truths have indeed real existence at a definite time in the mind of the being that recognizes or thinks them.

The word *proposition* [Satz], because of its origin from the verb "to posit' [setzen] does indeed remind one of an action, of something that has been posited by someone. But we must abstract from this in the case of truths in themselves. They are not posited by anyone, not even by the divine understanding.

§34 Judgment [Urteil]
There is a certain common constituent in the concepts which are designated by
the words 'to assert', 'to decide', 'to opine', 'to believe', 'to take for true' and similar words.
(a) Every judgment contains a proposition which is either true or false. In the first case the judgment is called correct, in the second incorrect,
(b) Every judgment has existence,
(c) A judgment does not have its existence by itself but only in the mind of some being which forms the judgment,

(d) There is an essential difference between a judgment and the mere thinking or representing of a proposition.

§46
Good order requires that we should first deal with the parts of propositions, namely ideas [***Vorstellungen***], then with propositions in general, then with true propositions, and finally with arguments [***Schliisse***] which are a certain kind of proposition stating that relations of deducibility [***Ableitbarkeit***] or ground and consequent [***Abfolge***] hold between other propositions. [59]

§48:
Anything that can be part of a proposition in itself, without being itself a proposition, I wish to call an *idea in itself,* or simply an *idea* or *objective idea.*
the combination of words 'Caius has wisdom' expresses a complete proposition. The word 'Caius' itself expresses something that can be part of a proposition… although it does not by itself form a proposition. This something I call an idea [Vorstellung].
Similarly, what is designated by the word 'has' and indicated by the word 'wisdom' I call ideas. [61]

[Distinguishes "*subjective* or *mental* ideas" from "objective ideas."]
An objective idea does not require a subject but subsists *[bestehen*]*,* not indeed as something *existing,* but as a certain *something* even though no thinking being may have it; also, it is not multiplied when it is thought by one, two, three, or more beings, unlike the corresponding subjective idea, which is present many times. Hence the name 'objective'. For this reason, any word, unless it is ambiguous, designates only one objective idea, but there are innumerable subjective ideas which it causes. [62]
§ 48. *What the Author Means by an Idea in Itself, and an Idea which Someone has*
2. Anything that can be part of a proposition in itself, without being itself a proposition, I wish to call an *idea in itself,* or simply an *idea* or *objective idea.*
[Bob B: B follows Kant in starting his *definitions* with Sätze. He then *defines* Vorstellungen as *parts* of Sätze. This, even though he follows traditional organization of his *treatise*, starting with the parts rather than the significant wholes. The first exclusion means that Sätze are *not* a kind of *Vorstellung*. Will he later disavow this, to allow propositions to be parts of propositions? Or does he always treat inclusion of a Satz as actually inclusion of something related to it (a Vorstellung of it, i.e. having it as an object?).]

§49 **The *objects* to which a Vorstellung *refers*, or which it *represents*.**
meant nothing but the o*bject* to which the mental idea *refers.*
By object of an idea I mean that something (sometimes existing and sometimes non-existing) of which we say that the idea [representation] *represents* it, or *of* which it is a representation.

[Bob B: He assumes for now that Vorstellungen *do* have objects. Later §108, he will show how to extend his definitions, e.g. of incompatible attributes, to Vorstellungen without referents.]

§ 49
thus we should never think of ideas in themselves as being propositions in themselves, but only as being actual or possible parts of such propositions.
This does not mean that an idea cannot contain a whole proposition or even several of them as parts. For even complete propositions can be combined with certain other ideas in such a way that the whole which is thus formed does not assert anything unless further parts are added. Hence, such a whole will not be called a proposition, but a mere idea.
Thus, e.g. the words 'God is almighty' expresses a complete proposition which recurs in the following combination of words 'Knowledge of the truth that God is almighty'; but this new combination of words no longer expresses a complete proposition. However, a new proposition can be generated through a further addition, e.g. when we say 'Knowledge of the truth that God is almighty can give us much consolation'. Consequently, what is expressed by the words 'Knowledge of the truth that God is almighty' alone, is a mere idea, although one that has a complete proposition as a component.
[Bob B: It looks as though the view is that

i) Vorstellungen are parts of Sätze (§48),
ii) Sätze can also be parts of Vorstellungen, but is he also endorsing
iii) Sätze can *only* be parts of Sätze *by* being parts of Vorstellungen that are parts of Sätze?

That is not going to work for conditionals or conjunctions, or for arbitrary embedding of negations and conditionals and so on. He does not actually *say* (iii), though.
Q: Does he give us the tools to handle arbitrary embeddings?]
This later does *not* settle the issue:

§49 [Distinguishes Vorstellungen from their *objects*, Gegenstände]
The object to which an idea refers or (as it may be called for short) the object of an idea I wish to distinguish not only from the mental idea, but also from the idea in itself on which the latter is based. I want to say that whenever a mental idea has one, none or several objects, the corresponding idea in itself must also have one, none, or several objects. They must, indeed, have the same objects.
By object of an idea I mean that something (sometimes existing and sometimes non-existing) of which we say that the idea [representation] *represents* it, or *of* which it is a representation. [62-3]
..how important it is to distinguish an idea, subjective as well as objective, from its object. **An objective idea…is never something *existing***. The object, on the other hand, to which an idea refers can have existence (as in the present example Plato, Socrates, etc.).

However, hardly anyone will deny that in the same sense in which the idea 'philosopher' refers to the objects Socrates, Plato, etc., the idea 'proposition' refers to things that are called the Pythagorean theorem, the theorem of the parallelogram of forces, the law of the lever, etc. The only difference is that Socrates and Plato are existing things, while these propositions, as propositions in themselves, do not exist. [63]

To distinguish the *word* which is introduced to designate an idea from the idea itself is even more important than to differentiate between an idea and its object. A word is always a physical object. (It exists at a certain time and in a certain location.) It is always a combination of sounds or of written signs. But an idea in itself, as I have said, does not exist.

§50:
When I say of an idea that it has reference [*Gegenständlichkeit]* I mean nothing but that there are objects which are subsumed under it,
It seems indisputable to me that every, even the simplest, proposition is composed of certain parts, and it seems equally clear that these parts do not merely occur in the verbal expression as subject and predicate (as some seem to think), but that they are already contained in the proposition in itself.

B. here acknowledges that it is somewhat contrary to common usage to speak of ideas *[Vorstellungen]* which nobody has. But he is not aware that there is a more suitable expression, except perhaps 'concept* *[Begriff],*which he will reserve for ideas that are not intuitions *[Anschauungeri].*

§54 "One attribute which all ideas have in common is that they do not have real existence."

§55 "A second attribute which belongs to all ideas is that neither truth nor falsity can be attributed to them. Only complete propositions are true or false. By ideas, however, we mean parts of propositions which are not themselves propositions. Hence, neither truth nor falsity can be attributed to them."

§56 if the *mental idea* is composed of several clearly distinguishable parts, then there can be no doubt that the *idea in itself,* which forms the matter of this mental idea, must also consist of parts. Hence, ideas in themselves are sometimes composed of parts. The sum of the parts of which a given idea consists is usually called its *content.* Consequently every complex idea must undeniably have a content.
By the content of an idea we mean the sum of the components of which this idea consists, but not the way in which these parts are connected. It follows that andidea is not completely determined by its content, but that two or more different ideas can sometimes be formed from a given content. Thus the two ideas 'a learned son of an ignorant father' and 'an ignorant son of a learned

father are quite different, though they have obviously the same content. The sameholds of such ideas as '35' and '53', and others.

Bob B: ‘Content’ is mereological, but the *idea* (Vorstellung) has a structure, too.

NOTE 1: I have asserted that every idea in itself is composed of at least as many parts as we distinguish in the mental idea whose matter it is. By this I mean that the former may well have more parts than we clearly distinguish in the latter. It is true that we think a certain idea in itself, i.e. have a corresponding mental idea, only if we think all the parts of which it consists, i.e. if we also have mental ideas of these parts. But it is not necessarily the case that we are always clearly conscious of, and able to disclose, what we think.

§57 If some ideas are indeed complex, then it will be one of the aims of **logic** to discuss the more important kinds of complexity in ideas.

§58 3. B. suggests a second sense in which immediate and remote parts of ideas could be distinguished: if an idea contains a proposition, then the ideas in that proposition could be called remote parts of the idea. All other parts could be called immediate.

7. From the preceding it becomes clear that the constituents of an idea frequently occur in an order which is not arbitrary. If this order were changed, another idea would result, as I have said above (§ 56). It is evident that we must not think of this often essential order of the parts of an idea as a succession in *time.* For an idea in itself is nothing real, and hence we cannot say of its parts that they exist at the same time, nor that they follow one another at different times. It is different with *mental* ideas, which are something real.

§59 which I call the more exact sense, '*this* A' means roughly the same as 'this, which is an A

§61 Therefore, any idea, however complex it may be, even if it contains infinitely many parts (if that is possible), must have parts that do not allow further division. These simple parts cannot be propositions, since every proposition, taken as such, is complex. Since the only parts of ideas are either propositions or other ideas, the simple parts must be ideas. Thus it follows that there are simple ideas.

§ 63 I have no reason to believe that the idea of an object must be composed of the ideas of all the parts that must necessarily belong to the object so that the object falls under the idea.

§ 66

1. It has been noted before that for most, though not for all ideas, there exists something to which they refer. This something I have called their object or referent, using the word 'object' in its widest sense; in §49 I have already tried to convey what concept I connect with it.
2. Of ideas that have one or more objects I say that they have *reference.* On the other hand, of those that have no object corresponding to them I say that they *lack reference [gegenstandslos].* By indicating the objects to which a certain idea refers, we indicate the *range, extension,* or sphere of this idea.

§67 It is true that most ideas have some, or even infinitely many, referents.
Still, there are also ideas that have no referent at all, and thus do not have an extension.
The clearest case seems to be that of the concept designated by the word 'nothing'.

§ 70 "The expression 'empty' must not be connected with the content of these ideas. Thus one should not think of these ideas as without content, for all complex ideas have a certain content, namely certain constituents. They are called empty only with respect to their extension . . . it should be remembered that inconsistent ideas are not the only ones that are empty in this sense; there are many other ideas that do not have objects, although they are not inconsistent, for example the idea 'nothing'. The peculiarity of inconsistent ideas is that they lack objects because they attribute contradictory properties to any possible referent".

§ 72. *What the Author Means by Intuitions*
Among the ideas so far discussed the most important are these: singular ideas as far as extension is concerned, and simple ideas as regards content. Ideas that combine these two properties, i.e. that are simple and yet have only one object, would be even more remarkable. The question is whether there are such things.

As soon as we direct our attention upon the change that is caused in our mind by an external body, e.g. a rose that is brought before our senses, the *next* and *immediate* result of this attention is that the *idea* of this change arises in us. Now, this idea has an object, namely the change that takes place in our mind at that very moment, and nothing else. Thus, it has only one object and we can say that it is a *singular* idea. On this occasion other ideas, some of them no longer simple, are also produced by the continued activity of our mind; similarly, complete judgments are made, especially about the change itself that has just taken place. We say, for instance, 'this (what I just see) is the sensation or idea 'red''; 'this (what I just smell) is a pleasant fragrance'; 'this (what I just feel upon touching a thorn with the tips of my fingers) is a painful sensation', etc. It is true that in these judgments the ideas 'red', 'pleasant fragrance', 'pain', etc. have several objects. However, the ideas which occur in subject position and which we designate by the word 'this' are certainly genuine singular ideas (§ 68). For, by the 'this' we mean nothing but this individual change which takes place in us, and not a change that takes place elsewhere, no matter how similar it is to ours. Moreover, it is no less certain that all these ideas are also *simple.*

§ 73. *Concepts and Mixed Ideas*
1. Given that there are such ideas as I have just described under the name of intuitions, and that these ideas are quite important, it can hardly be doubted that there are ideas that are not intuitions and do not contain any intuition as constituents, and that they are important enough to deserve a special name. I call these ideas *concepts,* because I think that this word has been used with a very similar sense ever since intuitions and concepts were first contrasted. Thus the idea 'something' I wish to call a concept; it is not an intuition, since it has not one, but infinitely many, objects, and it does not contain any intuition as a constituent, for it is not complex at all.

§ 92. *Relations between Ideas with Respect to their Content*
1. In § 56 the content of an idea was defined as the sum of its components, irrespective of order. Hence there may be different ideas with the same content. They are contrasted with ideas that share no part at all. Simple ideas that have the same content must be identical, because simple ideas and their content are one and the same thing.
2. Some ideas share some, but not all, components. B. calls them 'kindred ideas'. The larger the number of common parts, the "closer" the kinship. Kinship and similarity must be carefully distinguished. *'A'* and 'non-*A'* are akin, but not similar. On the other hand, "the idea of moral goodness is often confounded with, and hence similar to, the ideas of public utility, honesty, etc., although it is likely that they have not a single common component."

§ 94. *Relations between Ideas with Respect to their Objects*
1. Concerning the objects themselves which are represented by ideas, it appears that two ideas either have common objects or they do not. Both cases are important enough to deserve a special name. I therefore call ideas that have one or several objects in common *compatible*, and those that have not even a single common object I call *incompatible.*

§ 95. *Special Kinds of Compatibility:* a. *Inclusion*
1. If the concept of compatibility is taken in the sense of the preceding section, then there are several species of this relation that deserve special consideration. If a pair of ideas *A* and *B* stand in the relation of compatibility to one another, then it may be the case that not only one, but all of the objects under *A* also fall under *B.* If it is not assumed that this relation is mutual, i.e. if it is undetermined whether *B* has some objects that do not fall under *A,* then I wish to call this relation between *A* and *B* a relation of *inclusion.* I wish to say that the extension of the idea *B,* or briefly that *B,* includes *A,* and I call *B* the including and *A* the included idea. Thus I wish to say that the idea 'man' is included in the idea 'inhabitant of earth', since every referent of 'man' also falls under 'inhabitant of earth'.

even if the ideas *A, B,...* have a common object, this does not mean that it is also common to the remainder *C, D,. . . .*
3. If a pair of attributive ideas *a* and *b* are compatible with one another, then their concreta *A* and *B* (§ 60) are also compatible. However, the converse does not hold. . . . For, if *a* and *b* are compatible

§ 96. b. *The Relation of Mutual Inclusion or Equivalence*
1. The definition of the concept of inclusion in the preceding section allows this relation to be mutual between two ideas *A* and *B. A* can be included in *B* and *B* in *A.* This is the case when all objects of *A* are objects of *B,* and all objects of *B* are also objects of *A.* Or, more briefly, if both ideas have precisely the same objects. This relation I call a *mutual* or *precise inclusion* or *equivalence,* and the ideas themselves I call *equivalent* ideas. The two ideas 'equilateral triangle' and 'equiangular triangle' are an example.

Scopes vs. Extensions
§97c
to be equivalent, it is not necessary that any one of the ideas *A, B, C, D,... ,* taken by itself, should be equivalent to any one of the *M, N, O, . . .*, nor is it necessary that the sum of the scopes of the ideas *A, B, C, D, . . .* should equal that of the ideas *M, N, O, . . .* The first can be seen from the preceding example. The second is shown in the following example. The two ideas 'member of the series 1, 2, 3, . . . 10' and 'member of the series 2, 3 , . . . 11' taken together are certainly equivalent to the following two 'member of the series 1, 2, 3, 4, 5' and 'member of the series 6, 7, . . . 11'. Yet the scope of the first idea is 10 and so is that of the second; thus the sum of both scopes is 20. But the scope of the third idea is 5, and that of the last one 6, with the sum of them merely 11. . . .
[Bob B: Counting extensions counts tokens, scopes count types.]

§ 105. *Some Theorems*
This section contains theorems that belong to §§ **93-104.** B. gives proofs which are here omitted.
1. If *X* is subordinate to both *A* and *B,* then *A* and *B* are compatible ideas.
2. If two ideas are incompatible, then there is no third idea which is subordinate to them both.
3. If two ideas are compatible, then all ideas higher than they are alsocompatible.
4. If two higher ideas are incompatible, then all ideas under them are also incompatible.
5. If two incompatible ideas *A* and *B* are compatible with *X,* then *X* is not subordinate to either of them.
6. If two ideas *A* and *B* overlap, then there is no idea *X* which is higher than *A* and lower than *B :* either *X* is higher than *B,* too, or overlaps *B.*
7. If *A* and *B* overlap, and *X* is lower than *A,* then *X* is either lower than *B,* or overlaps *B,* or is incompatible with *B.*

8. If *A* and *B* overlap, and *X* is incompatible with *A,* then *X* must be lower than, or incompatible with, *B*.
9. If *A* is higher than *B,* then *non-A* is lower than non-13.
10. No idea can be incompatible with both *A* and *non-A:* If it is incompatible with one, then it must be equivalent or subordinate to the other.
11. Two merely contrary ideas can be subordinate to the same higher idea.
12. Both of two contrary ideas can be incompatible with one and the same idea.
13. One and the same idea can be compatible with each of two contrary ideas, and also with each of two contradictory ideas.
14. If *B* is lower than *A,* then *A* and *B* are compatible.
15. If *A* is neither equivalent nor subordinate to non-B, then it must be compatible with B.
16. If *A* and *B* are compatible, then of the two ideas *A* and non-*B* and of the two ideas *B* and non-*B* neither includes the other.
17. Conversely, if inclusion holds between *A* and non-B, then *A* and *B* are incompatible.
18. If *A* and *B* both have referents, but neither includes the other, then *A* and non-*B*, *B* and *non-A* are compatible, and neither of *non-A* and non-*B* is subordinate to the other.
19. If *A* and non-*B* and *B* and *non-A* are compatible, then *A* and *B* are either incompatible, or they overlap.
20. If *A* and *B* are contraries, then *A* and non-*B*, and *B* and *non-A* are not equivalent; non-*A* and non-*B* are consistent in this case.
21. If *B* is lower than *A,* then non-*B* cannot be lower than *A,* except if *A* is an idea of greatest scope. If *B* is lower than *A,* then a contrary of *B* can be lower than *A*.
22. If *A* and *B* overlap, then it is possible that *non-A* is subordinate to *B*.
23. If the extension of two overlapping ideas *A* and *B* is smaller than the largest extension, then *non-A* and non-*B* also overlap.

§107 Opposing ideas

The definition of this concept, it seems to me, is the following: Let *A* be an idea which represents precisely a, and let *B* be an idea which represents precisely b, and let *p, q, r,. . .* be pure concepts. We shall call two objects a and b *opposed* to each other if there are some *p, q,r, .. .* such that with *A* they will form an idea [*A, p, q,r,.. .*] which represents precisely b and with *B* an idea [*B, p, q, r,...*] which represents precisely a.

[Bob B: This uses "pure concepts" (I assume, those without intuitions as parts) to define opposition of *objects*. This looks like a top-down order of definition.]

§ 108. *Ways in which the Relations Considered in* §§ 93 ff.*could be Extended to Ideas that do not have Referents*

According to their definitions, the relations between ideas that we have considered in § 93 if. cover only ideas that have referents. It is nevertheless certain that we apply, even in common language, several of those relations to ideas which represent no object, or even to those which *cannot* represent an object, since they assign certain contrary attributes to it.

In § 69, we discussed the concept of redundancy. This concept was originally defined in such a way that it could be applied only to ideas with referents. But we discovered a means for a suitable expansion of it by considering certain constituents of the given idea *i, j*... to be *variable.* This method can be applied in the present case too. We can at once extend all of the definitions mentioned in §§ 93-107 to ideas without referents, if we are only permitted to envisage certain of their parts as variable. For then it would only be necessary to consider the infinitely many new ideas that are generated from the given ones if the variable parts *i, j,* • . . are replaced by any other ideas. The same relation that holds between these new ideas, *whenever they have a referent,* we also attribute to the given non-referring ideas. It is understood that this is the case only *relative* to the parts *i, j,* . . . which are taken to be variables. In particular, we shall say that two non-referring ideas *A* and *B* are *equivalent* with respect to the variable parts *i,j,* . . ., if the ideas that are generated when we put any other ideas in the place of *i, j,*. . . are *equivalent* in the narrower meaning § 96 **whenever they have referents**. We shall say that *A* is *higher, B lower,* if the new ideas that are generated from *A* and *B* stand in the relation of subordination in the sense of § 97 whenever they have referents, etc. Accordingly, the two ideas 'mountain that is of gold' and 'gold that forms a mountain' shall be considered equivalent with respect to the variable parts 'mountain' and 'gold'. For all ideas that are generated if we replace these two parts by whatever other parts are equivalent in the sense of § 96, i.e. have the same objects, if they have any objects at all.
[Bob B: Here "have the same objects if they have any objects at all" is parallel for Vorstellungen to "are true, if they have any truth-value at all". The asymmetric, implication-like relation has one set of substituends giving B an object if it gives A an object.]

§ 124. *Every Proposition can be Viewed as Part of another Proposition, even as Part of just an Idea*

Propositions can be indefinitely compounded, i.e. any of them can be constituents of other propositions and ideas. For example, we can form a class of propositions *A* and *B;* in thinking this class we have an *idea,* and if we assert something about this class, we generate a proposition of which *A* is a constituent.
[Bob B: He presumably thinks that conjunction is a way of operating on that class of propositions, as well as others.
Like the copula, it is just more content. *Form* is gotten only from variation.]

§ 127. *Parts which the Author Takes all Propositions to have*

Closer consideration shows that *all* propositions have three parts, a subject idea, the concept of having, and a predicate idea, as indicated in the expression *'A* has *b'*. In the sequel, a number of propositions, or rather kinds of propositions, will be shown to have these parts, although this is often obscured by their linguistic form.

§ 133. *Conceptual and Empirical Propositions*
No matter what anybody's theory concerning the parts of propositions may be, he can hardly deny that there are propositions, even true propositions, which consist entirely of pure concepts, without containing any intuitions [*Anschauung*]. The following propositions are obviously of this sort: 'God is omnipresent', 'gratitude is a duty', 'the square root of *2* is irrational', etc. We shall see in the sequel how propositions of this kind differ essentially from propositions which contain intuitions, especially if they are true. Hence I believe that a special designation for them is indispensable for the purposes of science, and I shall call them *conceptual propositions* or *conceptual judgments* and, if they are true, *conceptual truths.* All other propositions will contain one or several intuitions and may for this reason be called *intuitive propositions*, also *empirical* or *perceptual* propositions, etc. I shall therefore call propositions like 'this is a flower', 'Socrates was born an Athenian', empirical, since each of them contains at
least one, perhaps several, intuitional ideas.

§147 It cannot be denied that every given proposition is either true or false and never changes: either it is true forever, or false forever, except if we change some *part* of it, and hence consider no longer the same but some other proposition (cf. § 125). We do this frequently without being clearly aware of it; this is one of the reasons why it seems as if the same proposition could sometimes be true and sometimes false, depending on the different times, places or objects to which we relate it. Thus we say that the proposition 'The keg of wine costs 10 Thaler' is true at this place and time, but false at another place or time.
These examples show that we often take certain ideas in a given proposition to be variable and, without being clearly aware of it, replace these variable parts by certain other ideas and observe the truth values which these propositions take on.

Let us, for example, consider the proposition 'The man Caius is mortal', and let us envisage the idea 'Caius' as arbitrarily variable; hence we put in its place any other idea, e.g. 'Sempronius', 'Titus', 'rose', 'triangle', etc. If we do this, it becomes obvious that the new propositions which are thus generated are all true, without exception, as long as they have reference, i.e. as long as their subject-idea has a referent. For, if we replace 'Caius' by ideas which designate real men, e.g. 'Sempronius', 'Titus', we always get true propositions**. On the other hand, if we take some other idea, e.g. 'rose', 'triangle', then we obtain a proposition which not only lacks truth, but which does not even have reference.**

[Bob B: This seems to be how he distinguishes substituends that do not “fit” the position they are substituted into: the proposition that results does not have a *reference*, or, therefore, a truth-value.
But what is the ‘reference’ of a proposition, if not its truth-value?]

Most propositions by far are such that the propositions which evolve from them are neither all true nor all false, but some of them are true and others are false. Here the question arises *how many* are true and how many false, or what the relation is between one set and the other (or the total set). If we allow a replacement of the ideas which we consider variable by any other idea whatever, then the totality of the true as well as false propositions which can be generated from a given proposition will be infinitely large.

I therefore want to give a special name to the concept of the relation of all true propositions to the total of all propositions which can be generated by treating certain ideas in a proposition as variables and replacing them with others according to a certain rule. I wish to call it the *satisfiability [Gültigkeit*] of the proposition.

[Bob B: This is usually translated as "validity," in Kant, Frege, and Tarski.]

The degree of satisfiability of a proposition is expressed by the relation between the number of the true propositions and the total number of propositions which are generated when certain ideas contained in the original proposition are considered variable and exchanged for others according to a certain rule. The degree of this satisfiability can then be represented as a fraction, where the numerator is to the denominator as the number of true propositions to the total number of propositions.

It is evident that the satisfiability of a proposition will vary, depending on which and how many of its ideas are considered variable. Thus, for instance, the proposition 'This triangle has three sides' remains true so long as only the idea 'this' is considered variable, provided that all ideas put in its place will generate referring propositions. However, if in addition to 'this', we also consider the idea 'triangle' or, instead of both, the idea 'side' as a variable, then the **degree of satisfiability of the proposition** will turn out to be entirely different. **Thus, in order to give a proper estimate of the degree of satisfiability of a proposition, there must always be an indication which of its ideas are to be considered variable**.

§ 148. *Analytic and Synthetic Propositions*

1. I showed in the preceding section that there are universally satisfiable as well as non-satisfiable propositions, given that certain of their parts are considered variable. It was also shown that propositions which have either of these properties on the assumption that *i, j, . . .* are variable, do not retain this status if different or additional ideas are taken as variable. It is particularly easy to see that no proposition could be formed so as to retain such a property if *all* its ideas were considered variable. **For if we could arbitrarily vary all constituent ideas of a proposition, we could transform it into any other proposition whatever, and thus could turn it into a true, as well as a false, proposition**.

If a proposition contained even a single idea which could be arbitrarily changed without altering the truth or falsity of the proposition; i.e., the propositions which could be obtained from it through the arbitrary alteration of this one idea would either all be true or all false, provided only they have reference. Borrowing this expression from Kant, I allow myself to call propositions of this kind analytic. All other propositions, i.e. all those which do not contain any ideas which can be changed without altering their truth or falsity, I shall call synthetic.

For example, I call the following propositions

analytic: 'A depraved man does not deserve respect' and 'A man may be depraved and still enjoy continued happiness'. The reason for this is that both contain a certain idea, namely 'man', which can be exchanged for any idea whatever, for instance 'angel', 'being', etc., yet the former remains always true, the latter always false, **provided only that they continue to have reference**.

[Bob B: This last notion is one it is important to get clear about, since it defines substitutions of the wrong category.]

The following are some very general examples of analytic propositions which are also true: *'A* is *A',* 'An *A* which is a *B* is an *A,*' 'An *A* which is a *B* is a *B*,' 'Every object is either *B or* non-*B*', etc. Propositions of the first kind, i.e. propositions cast in the form *'A* is *A' or 'A* has (the attribute) *a'* are commonly called *identical* or *tautological* propositions.

3. The difference between the last mentioned analytic propositions and those under no. 1 lies in the following: In order to appraise the analytic nature of the propositions under **2,** no other than logical knowledge is necessary, **since the concepts which form the invariable part of these propositions all belong to logic**. On the other hand, for the appraisal of the truth and falsity of propositions like those given in no. 1 a wholly different kind of knowledge is required, since concepts alien to logic intrude. This distinction, I admit, is rather unstable, as the whole domain of concepts belonging to logic is not circumscribed to the extent that controversies could not arise at times. Nevertheless, it might be profitable to keep this distinction in mind. Hence propositions like those in no. 2 may be called *logically* analytic, or analytic in the *narrower* sense; those of no. 1, analytic in the *broader* sense.

Schmidt has reminded us that Crusius *(Weg zur Gewißheit,* Leipzig 1747, § 260), already saw the difference between analytic and synthetic judgments in the same way as Kant.

CHAPTER 3

Distinctions among Propositions which are Based upon their Relations to each other

[Bob B: This section is the most important one in the book so far. It is worth studying all alone.]

§ 154. *Compatible and Incompatible Propositions*

1. The most important relations among propositions come to light when we follow the method introduced in § 147, and envisage certain ideas contained in them as variable; we then consider the new propositions which are generated, if these ideas are replaced by any other ideas whatever, and observe what truth values they take on.

2. We know already that **almost any proposition in which we replace certain ideas by different, arbitrarily chosen, ideas will sometimes turn into a true, and sometimes into a false, proposition**.

Let us compare several propositions *A, B, C, D,* . . . and consider as variable certain ideas *i, j,* . . ., which they have in common. The question arises whether there are any ideas which can be put into the place of *i, j,* . . . and which are of such a nature that they make *all* of the above propositions true *at the same time.* If this question must be answered in the affirmative, then I wish to call the relation between propositions *A, B, C, D,* .. . a relation of *compatibility,* and the propositions *A, By C, D,*.. . will be called *compatible* propositions. If the question must be answered in the negative, i.e. if there are no ideas whose substitution for *i,j,* . . . will make the propositions *A, B, C, D,*. . . all true, I say that they stand in a relation of *incompatibility,* and the propositions themselves I call *incompatible.*

[Bob B: This is joint *satisfiability* understood in a way this both broadly substitutional and not narrowly substitutional in a way that restricts that notion to syntax or lexicon. For it is the *objective* Sätze and Vorstellungen that are at issue, not the *mental* or *subjective* representatives of them, nor the linguistic expressions of them that are to be regarded as having variable parts.]

3. It follows from the above definition that the relation of compatibility and incompatibility holds mutually (between propositions).

With ideas, the crucial question was whether or not a certain object is indeed represented by them; the corresponding question for propositions is whether or not they are true. Just as I have called ideas compatible or incompatible with each other, depending on whether or not they have certain referents in common, so I call propositions compatible or incompatible, depending on whether or not there are certain ideas which makes all of them true.

5. A class of propositions *A, B, C, D,* . . . can appear as either compatible or incompatible, depending upon which ideas are considered variable. Hence the two propositions 'a lion has two breasts' and 'a lion has two wings' are compatible if we envisage the idea 'lion' as variable : if we replace it by the idea 'bat', both propositions become true. But if the idea 'two' is the only one that may be altered, then the two propositions present themselves as incompatible, since there is no idea that may be put in its place which makes both propositions true.

It is quite obvious that, **if we are permitted to increase indefinitely the number of variable ideas in a given class of propositions, then these propositions will always appear compatible. For, if we can change arbitrarily many, perhaps all, of the ideas that occur in a given proposition, then we can change every proposition into any other one, hence doubtlessly also into a true proposition.**

Thus, if we say of a given class of propositions *A, B, C, D,. . .* that they are compatible or incompatible, we must indicate, for the sake of precision, *in what respect,* i.e. in relation to what variable ideas *i, j, . . .* this is meant.

6. All truths are compatible with each other no matter which oftheir ideas are considered variable: they are already true by virtue of the ideas of which they originally consist. 7. Hence of any given set of incompatible propositions at least one must be false, but several, even all of them, may be false.
8. In a set of compatible propositions, there may also be false ones, even all of them may be false. The fact that certain propositions are false, given the ideas of which they originally consist, does not prevent them from becoming true if certain other ideas are introduced. It is obviously necessary, however, that each of these false propositions should contain at least one of the ideas which are considered variable, since otherwise it would not change and hence not turn into a truth.

11. If certain propositions are compatible with each other with respect to the ideas *i, j, . . .*, then they are also compatible with respect to the more inclusive set *i, j, k, l, .. .* which contains the former. If certain propositions are incompatible with respect to the larger set of ideas *i, j, k, l,. . .*, then they are also incompatible with respect to the smaller set *i, j. . ..* Conversely, it does not follow that, if a number of propositions are incompatible with respect to ideas *i,j, . . .*, they are also incompatible with respect to *i,j,* and if they are compatible with respect to ideas *i,j, k, l* , . . . , it does not follow that they are also compatible with respect to *i, _l....*

12. From the fact that propositions A, B, C, D, . . . as well as propositions G, H, I, K. . . are compatible with propositions M, N, O, . . . with respect to ideas i, j, . . . it does not follow that propositions A, B, C, D, . . . and G, H, I, K, . . . are compatible with each other with respect to these same ideas. It could very well be the case that there are certain ideas whose substitution for i, j, . . . makes propositions A, B, C, D, . . . and M, N,O, . . . true at the same time, while there are certain other ideas which make G, H, I, K, . . . and M, N, O, . . . true. But there may be no ideas at all which make A, B, C, D, . . . and G, H, I, K, . . . true at the same time. For example, both propositions 'all A are B' and 'no A is B' are compatible with the proposition 'all A are C' with respect to the three ideas A, B, C; nonetheless, the first two propositions are not compatible with each other with respect to the same ideas.
13. Similarly, it does not follow that if propositions *A, B, C, D, . . .* as well as *G, H, I, K,. . .* are incompatible with propositions *M, N, O, . . .* with respect to certain ideas *i, /* , . . . , then *A, B, C, D, . . .* and *G, H, I, K,. . .* are incompatible with each other.. . .

16. All propositions whose predicates are viewed as variable are compatible with each other no matter what their subjects are, so long as the latter have reference. For, if each of these propositions has a distinct predicate idea, which differs from that of the others, but is viewed as

variable, then it will be easy to give each of these propositions a predicate which makes it true. All we have to do is find for each of them severally the idea of an attribute which all objects represented by its subject have in common. But if some or all of them have one and the same predicate idea, then we have to find an idea of an attribute which the totality of objects represented by their various subjects have in common. Such attributes can always be found, since even the most divergent objects have certain attributes in common.

§ 155. Special Kinds of Compatibility :

a. The Relation of Deducibility

Let us, to begin with, consider the relation of *compatibility.*

2. If we assert that certain propositions *A, B, C, D,. . . M, N,* 0, . . . stand in the relation of compatibility with respect to ideas *i, j,.. .*, then we assert, according to the above definition, no more than that there are certain ideas whose substitution for *i, j,.. .* turns all of those propositions into true ones.

Let us consider, first of all, the case that among the compatible propositions *A, B, C, D, . . . M, N, O, . . .* the following relation obtains : all ideas whose substitution for the variable ideas *i,j, . . .* turns a certain part of these propositions, namely *A, B, C, D, . . .* into truths, also have the characteristic of making a certain other part of these propositions, namely *M. N, O,. . .* true.

I wish to give the name of *deducibility [**Ableitbarkeit**]* to this relation between propositions *A, B, C, D, . . .* on one hand and *M, N, O, . . .* on the other. Hence I say that propositions *M, N, O, . . .* are *deducible* from propositions *A, B, C, D, . . .* with respect to variable parts *i, j, . . .* , if every class of ideas whose substitution for *i, j, . ..* makes all of *A, B, C, D,. . .* true, also makes all of *M, N, O, . . .* true. Occasionally, since it is customary, I shall say that propositions *M, N, O,. . . follow,* or can be *inferred* or *derived,* from *A, B, C, D. . . .* Propositions *A, B, C, D,... I* shall call the *premises, M, N, O,. . .* the *conclusions.* Finally, since this relation between propositions *A, B, C, D, . . .* and *M, N, O,. . .* has great similarity to the relation between including and included ideas, I shall allow myself to call propositions *A, B, C, D, . . . included*, and *M, N, O, . . . including* propositions.

26. Let proposition *M* be deducible from premises *A, B, C, D, . .* with respect to ideas *i, j. . . .* If *A, B, C, D, . . .* are such that none of them, nor even any of their parts, may be omitted, with *M* still deducible from the remainder with respect to the same ideas *i,j, . . .,* I call the relation of deducibility of proposition *M* from *A, B, C, D, . . . irredundant [genau bemessen], precise,* or *adequate.* A relation of deducibility that is not irredundant is said to be *redundant.*

It also seems to me that the relation of deducibility should not be confounded with what I shall later call the ground-consequent relation *[Abfolge];* this relation, I believe, does not hold between propositions in general, but only between *truths.*

12. If two propositions *'X* has *a'* and *'X* has *b'* have the same subject, which is to be the only variable idea in them, then they are equivalent if the ideas *A* and *B* are equivalent, otherwise not. For, if ideas *A* and *B* are equivalent, hence if every object of one of them also falls under the other, then every idea whose substitution for *X* makes one proposition true, must also make the other true. But if *A* and *B* are not equivalent, then there is some object which falls under one of them, e.g. *A,* which does not fall under the other, *B.* **There will then also be an idea which applies exclusively to this object**. Let this idea be *A'*. If we replace *X* by the idea *A'* then the proposition *'X* has *a',* but not the proposition *'X* has *b'* becomes true.
[Bob B: the assumption that if there is an object, then there is an objective Vorstellung that has that object as its sole referent, is doing a lot of work in getting Tarskian model-theoretic conclusions be drawn. Is there a distinctive kind of conceptual realism at play, downstream from Kant?]

§ 157. c. *The Relation of Subordination*
1. Let us assume that a relation of deducibility between propositions *A, B, C, D,* . . . and *M, N, O,* . . . is not reciprocal, as in the preceding section, but holds only in one direction, e.g. such that propositions *M, N, O,* . . . are deducible from *A, B, C, D,* . . . but not conversely, with respect to certain variable ideas *i, j,* . . . or (what comes to the same), if every class of ideas whose substitution for *i, j,* . . . makes propositions *A, B, C, D* . . . true, also makes propositions *M, N, O,* . . . true, but not conversely. In such a case we call the relation between propositions *A, B, C, D,* . . . on one hand and *M, N,* O , . . . on the other a relation of *subordination.*
Because of the similarity of this relation with the relation between ideas discussed in § 97, I wish to call propositions *A, B, C, D,* . . . the *subordinated* or *lower,* or, if this is objectionable, propositions of *smaller degree of satisfiability* or *more limited* propositions, or propositions which ***claim more***. On the other hand, I call propositions *M, N, O,*. . . *superordinated,* or *higher* or *propositions* with a ***greater degree of satisfiability***, or propositions which ***claim less***.

2. If propositions *M, N, O,* . . . are unilaterally deducible from propositions *A, B, C,*. . . with respect to ideas *i,* _/,... then there are always certain propositions which are compatible with *M, N, O,* . . . without being compatible with *A, B, C,* . . . with respect to the same ideas *i, j*. . . .
Bob B: Compare with *A,B,C* implying *M,N,O* in case everything incompatible with the conclusions is incompatible with the premises (but not necessarily *vice versa*).

If propositions *M, N, O,* . . . are unilaterally deducible from propositions *A, B, C, D,* .. and propositions *R, S,T,* . . . are also unilaterally deducible from propositions *M,N,0,* . . . with respect to the same ideas *i,j,* . . ., then propositions *R, S, T,* . . . are unilaterally deducible from propositions *A, B, C, D,* . . . with respect to the same ideas.
That *R, S, T,* . . . is deducible from *A, B, C, D,* . . . follows from § 155, no. 23, and that this deducibility is merely unilateral follows from the fact that **there are certain ideas which make**

all of *R,S, T,* . . . true without making all of *M, N, O,* . . . true, and these cannot make all of *A, B, C, D,* . . . true, for if they did, *M, N, O,* . . . would also have to become true.

4. If two propositions *'X* has *a'* and *'X* has *b'* have one and the same subject, which is to be their only variable part, then they stand in the relation of independence if, and only if, the ideas *A* and *B* are **linked**. For, if ideas *A* and *B* are linked, then there are objects which fall under both of them. Hence if we replace *X* by an idea which applies exclusively to such objects, both propositions will become true. But there are also objects which fall under one, but not under the other, of these ideas, and an idea which applies exclusively to such objects will make one proposition true but not the other. But if the ideas *A* and *B* are not linked, then it is either the case that one or both of them are non-referring, and then one or both of these propositions are formally false; or else the ideas *A* and *B* are incompatible with each other and then, according to § 155, no. 18, so are the propositions; or one of these ideas is subordinate to the other and then, according to § 155, no. 36, one of the two propositions is deducible from the other.

§ 159 Similar distinctions can be made with the relation of *incompatibility.* If we say of several propositions *A, B, C, D,* . . . merely that they stand in the relation of incompatibility to each other with respect to ideas *i,j,* . . ., then all we are saying is that there are no ideas whose substitution for *i, j,* . . . will make all of the propositions *A, B, C, D,* . . . true together. But by saying that *A, B, C, D,*. . . are incompatible with each other, we are not claiming that there may not be some of them, e.g. *A, B,* . . ., or *B, C,* . . . which are made true through certain common ideas, without, respectively, *C, D,* . . . or *A, B,* . . . becoming true also.
[Bob B:
Defining *incompatibility-exclusion* in parallel to the definition of *implication-consequence*:]

In § 155 we considered classes of compatible propositions *A, B, C, D,* . . . *M, N, O,* . . . and asked the question whether there are not some of them, *A, B, C,* .. ., which are of such a nature that every class of ideas whose substitution for the variables *i, j,*. . . makes all of them true, will also make one or several others *M, N, O,* . . . true.
Let us now ask whether among several incompatible propositions *A, B, C, D,* . . . and *M, N, O,* . . . there may not **be some *A, B, C,* . . . which are of such a nature that every class of ideas whose substitution for *i,* _/',... makes all of them *true*, will make certain others, *M, N,0,* . . . *false*.** If this is the case, then the relation of the propositions *M, N, O,* . . . to the propositions *A, B, C,* . . . is the exact opposite of the relation which we have previously called *deducibility.*
I wish to call it the relation of *exclusion* ; I shall say that one or several propositions *M, N, O,* . . . are *excluded* by certain others *A, B, C,* . . . with respect to variable ideas *i, j,* .. . if every class of ideas whose substitution for *i, j,* . . . makes all of *A, B, C,* . . . true, makes all of *M, N, O,* . . . false. *A, B, C,* . . . I call *excluding* propositions, *M, N, O.*. . . *excluded* propositions. Such a relation of exclusion holds, for example, between the propositions *'A* is *B'* and *'B* is *C'* on one hand and the proposition 'No *C* is *A'* on the other, if *A, B, C* are envisaged as the only variable

ideas. For every class of ideas which make the first two propositions true will make the third one false. Hence I call the first two propositions excluding, and the last proposition excluded by them.
Bob B: This is a ***bilateral* account**, on Bolzano's part, very close to our pragmatic version of incompatibility.
Also, the parallelism between the definition of implication and that of incompatibility is like that in our version.

3. It can also be the case that the relation of exclusion holds mutually and with respect to the same ideas i,j, . . . between propositions A, B, C, . . . and M, N, O. . . . In this case every class of ideas which make all of A, B, C, . . . true will make all of M, N, O, . . . false, and every class of ideas which will make all of M, N, O, . . . true will make all of A, B, C,. . . false. We can properly call this relation between propositions A, B, C, . . . and M, N, O, . . . a relation of mutual exclusion.
[Bob B: So the other notion of exclusion was *not* symmetric?]

6. Among propositions which stand in the relation of contradiction to each other there can be no proposition which is formally true or formally false with respect to the same variable ideas. For, according to our definition, it must be possible to make any of a set of contradictory propositions true as well as false.

§ 162. *The Relation of Ground and Consequence* [**Abfolge**]
1. There is an important relation which holds among truths, namely that some of them are related to others as *grounds* to *consequences.*

the relation of deducibility can also hold among false propositions in such a way that the truths which are generated by replacing certain variable ideas by others always stand in the ground-consequent relation to each other. This holds, for example, of the following two propositions: 'it is warmer in location *X* than in location Y' and 'the thermometer stands higher in location *X* than in location *Y'*, provided that the ideas *X* and *Y* are envisaged as the only variable ones. **It is obvious that both these propositions can become false if we can replace *X* and *Y* by arbitrary ideas**. But as soon as we choose two ideas which make the first proposition true, the second one also becomes true, and, moreover, the first stands to the second as ground to consequence. It must be noted, however, that the latter does not always hold where there is a relation of deducibility. For example, the propositions which we just considered are mutually deducible from each other: the proposition'the thermometer is higher in Xthan in *Y'* is deducible from the proposition 'it is warmer in *X* than it is in *Y'* ; but conversely, the proposition 'it is warmer in *X* than it is in *Y'* is also deducible from the proposition 'the thermometer is higher in *X* than it is in *Y'.* Still, nobody will think for a moment that, when they are both true, the last proposition can be the ground and the other proposition its consequence.

[Bob B: Ulf's "certainty"? Both downstream from Aristotle.]

§§ 164—167 have the purpose of bringing into normal form *('A* has *b')* propositions which state relations between other propositions. For example, if propositions *M, N, O,* . . . are deducible from *A, B,* C, . . . with respect to *i, j,* . .the following normal-form proposition will state that relation : 'Every class of ideas whose substitution for *i,j,* . . . in propositions *A, B, C, ... M, N, O,* ... makes all of the propositions *A, B,* C, . . . true—has— the attribute of also making all of the propositions *M, N,* O, . . . true'. **B. calls propositions of this form *arguments* [Schlüsse], He points out that not all actual arguments state which ideas are to be considered as variable, but it is always implied that some of them are**.

§ 170. *Propositions Whose Linguistic Expression has the Form*
'Nothing has attribute b'
After giving some examples such as 'nothing is completely perfect' B. draws attention to the ambiguity of such sentences as 'Nothing is better than this medicine'; he then analyses expressions of the form 'Nothing has property *V* in this way: **'the idea of an object which has property *b* does not have a referent'**.

§ 179. *Propositions with 'if' and 'then'*
Every reasonably well developed language contains rather common modes of expression involving *if* and *then* ; consider, for example, the following proposition: 'if Caius is a man, and all men are mortal, then Caius is mortal'. I have already stated (§ 164) that **we use this form in order to express a relation of deducibility of a certain proposition from one or several other propositions. But I do not think that every time we use ' if ' and 'then' we want to indicate the presence of certain ideas which are envisaged as variable and may be replaced by any other ideas without destroying the truth of the proposition**.

NOTE: The following remark is more grammatical than logical: we use an *indicative* verb in connection with 'if-then' (if *A* is the case, then *M* is the case) if we want to leave it entirely undecided whether the compared propositions *A* and *M* are true or false; on the other hand we use the *subjunctive* ('if *A* were the case, then *M* would be the case), if we want to indicate that propositions *A* and *M* are false as stated.

§ 182 **6. These definitions should make it obvious what I consider to be the sense of sentences which state a necessity, possibility, or contingency. Thus I believe, for example, that the assertion 'God exists with necessity' has no other sense than 'the proposition that God exists is a purely conceptual truth'. 'Every effect must necessarily have its cause' I take to mean nothing but 'it is purely conceptual truth that every effect has its cause', etc.**

§ 196

[T]rue propositions, just as propositions in general, do not have real existence. Hence it is not proper to say that certain truths (i.e. purely conceptual truths) have eternal existence, as if we wanted to say of others that they are perishable, or that something has stopped being true or will become true only in the future.

all *true* propositions must have an object with which they deal, though perhaps this does not hold for propositions in general. Hence **all true propositions must contain an idea which refers to this object, so that the so-called subject idea (or basis) must in all true propositions be a proper referring idea** (cf. § 66)

a. To the objection that there must be truths whose subject is not a mere idea, but a complete proposition, we reply that this could be the case at best with truths that make an assertion about a whole proposition. If the object about which we make a statement is itself a proposition, then (it might be thought) the subject of our judgment is not a mere idea, but a complete proposition, namely the proposition of which we then speak. But we will give up this opinion as soon as we realize the following: in order to say correctly that a certain proposition deals with a certain object it is necessary, not that this object itself, but an *idea* of it be a component of our proposition.
Thus, for example, when we make a judgment about this house, the proposition must contain an idea referring to this house; similarly, **every truth which deals with a complete proposition must necessarily contain an idea of this proposition as a part, and this idea must be the subject idea.** Hence **there is no doubt that in truths of this kind, too, the basis is an idea, indeed a referring idea**.
b. The matter is similar with those truths which seem to be concerned with so-called *imaginary objects* such as 'there is no round square'. We already know from § 137 that propositions of this kind mean merely this: 'the idea of a round square does not have a referent.' Hence its subject is again a referring idea, **for only the idea 'round square' is non-referring; the idea of this idea (which is the subject of the proposition) is a referring idea. Its referent is that first idea**.

3. Just as the subject of every true propositions must be a proper referring idea, so the predicate part must be a proper attributive idea.

§ 530. *Proofs by Reduction to Absurdity*
This section offers procedures for turning any proofs that appeal to *false* propositions (such as proofs by absurdity) into proofs that include only true propositions.
This is parallel to a claim of Frege's.

§ 89. *Affirmative and Negative Ideas*
1. The concept indicated by the word 'not' [non, *Nicht*] is simple. Concepts of which it forms a part are called negative in the wider sense.

2. Negative concepts in the narrower sense are those in which the negation occurs an odd number of times in immediate succession.
[Bob B: This is a *really* disappointing section. What *is* that simple concept? We must look elsewhere.]

§ 123. *Every Proposition Necessarily Contains several Ideas*
Every proposition is complex, ideas being its parts. Even apparently simple propositions such as 'come!' contain the concept of coming and of a certain obligation. Likewise the Latin *'sum'* contains the concept of existence and the idea 'I' as subject. Thus every proposition contains several ideas and may be called complex. " I shall call the sum of all its immediate and remote parts its *content.* Hence the proposition 'God has omniscience' is composed of the ideas 'God', 'has' and 'omniscience'. The sum of these three ideas I call the content of the proposition. The idea 'omniscience' is itself complex, being the idea of knowledge that extends over all truths. Thus it contains the ideas 'knowledge', 'truth', and others, all of which are considered parts of the content of the above proposition. Hence the content of a proposition stands to the proposition in the same relation in which the content of an idea stands to the idea (cf. § 56)."
In a note, B. points out that he takes the copula of a proposition to be part of its *content*, while most other logicians consider it part of the *form*.

§ 124. *Every Proposition can be Viewed as Part of another Proposition, even as Part of just an Idea*
Propositions can be indefinitely compounded, i.e. any of them can be constituents of other propositions and ideas. For example, we can form a class of propositions *A* and *B;* in thinking this class we have an *idea,* and if we assert something about this class, we generate a proposition of which *A* is a constituent.

§126
Now the idea in a proposition which represents the objects which the proposition is about I wish to call its *object idea, subject idea,* or *basis.*
[Bob B: So every Satz contains a special Vorstellung, its *object* Vorstellung, or *basis*. Perhaps *these* are what are combined when propositions are parts of other propositions?]

[Bob B: Does B think:
a) If you plug a Vorstellung into a place in a Satz that is transcategorial, it yields a Satz that is false—and nothing stronger? Or
b) That for some 'values' of 'variables', there just *is no* Satz that is the result of substituting A for i in S?
I think (b). When considering some parts of a Satz S as replaceable or variable, we can only look and see whether there are Sätze related to it by having something else in place of part *i* of S's *content* (the content is the unstructured *set* of all its parts). Then we can look, for the variants of

S (at *i*) that *do* exist (not in B's sense of "exist"), and ask whether all of them that are *true* also lead to true versions of T, when the same substitutions are made for *i* and the variants of T exist. If some *true* i-variant of S does not correspond to *any* variant of T, then it does not correspond to any *true* variant of T, and so the implication of T by S at *i* fails.

There is a twist to this: propositions which are not only false, but lack a referent—that is, their subject lacks a referent:]

§147
Let us, for example, consider the proposition 'The man Caius is mortal', and let us envisage the idea 'Caius' as arbitrarily variable; hence we put in its place any other idea, e.g. 'Sempronius', 'Titus', 'rose', 'triangle', etc. If we do this, it becomes obvious that the new propositions which are thus generated are all true, without exception**, as long as they have reference, i.e. as long as their subject-idea has a referent**. For, if we replace 'Caius' by ideas which designate real men, e.g. 'Sempronius', 'Titus', we always get true propositions. On the other hand, if we take some other idea, e.g. 'rose', 'triangle'**, then we obtain a proposition which not only lacks truth, but which does not even have reference.**

§154
It is also obvious that there is great similarity between this relation among *propositions* and the relation between *ideas* which I have given the same name in § 94, especially if the amplification in §108 is taken into account. With ideas, the crucial question was whether or not a certain object is indeed represented by them; the corresponding question for propositions is whether or not they are true. Just as I have called ideas compatible or incompatible with each other, depending on whether or not they have certain referents in common, so I call propositions compatible or incompatible, depending on whether or not there are certain ideas which makes all of them true.

It is quite obvious that, if we are permitted to increase indefinitely the number of variable ideas in a given class of propositions, then these propositions will always appear compatible. For, if we can change arbitrarily many, perhaps all, of the ideas that occur in a given proposition, then we can change every proposition into any other one, hence doubtlessly also into a true proposition. Thus, if we say of a given class of propositions *A, B, C, D,. . .* that they are compatible or incompatible, we must indicate, for the sake of precision, *in what respect,* i.e. in relation to what variable ideas *i, j, . . .* this is meant.