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Diagrammatic reasoning in Frege's *Begriffsschrift*

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Abstract In Part III of his 1879 logic Frege proves a theorem in the theory of sequences on the basis of four definitions. He claims in *Grundlagen* that this proof, despite being strictly deductive, constitutes a real extension of our knowledge, that it is ampliative rather than merely explicative. Frege furthermore connects this idea of ampliative deductive proof to what he thinks of as a fruitful definition, one that draws new lines. My aim is to show that we can make good sense of these claims if we read Frege's notation diagrammatically, in particular, if we take that notation to have been designed to enable one to exhibit the (inferentially articulated) contents of concepts in a way that allows one to reason deductively on the basis of those contents.

Keywords Diagrammatic reasoning · Frege · Ampliative proof

In Part III of his 1879 logic (Frege 1879) Frege proves a theorem in the theory of sequences on the basis of four definitions. He claims in *Grundlagen* that this proof, despite being strictly deductive, constitutes a real extension of our knowledge, that it is ampliative rather than merely explicative: "From this proof it can be seen that propositions that extend our knowledge can have analytic judgments for their content" (Frege 1884, §91). Frege furthermore connects this idea of ampliative deductive proof to what he thinks of as a fruitful definition, one that "[draws] boundary lines that were not previously given at all": "What we shall be able to infer from it [a fruitful definition], cannot be inspected in advance . . . the conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic" (Frege 1884,

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§88).[1] I will suggest that we can make good sense of these claims if we read Frege's notation diagrammatically.[2]

1 Background

The ancient Greek mathematical practice of demonstrating using a drawn diagram provides the paradigm of diagrammatic reasoning: one begins by drawing a diagram according to certain specifications and then one reasons through the diagram to the desired conclusion. And as I have argued elsewhere (Macbeth 2010), the practice works in virtue of a very distinctive feature of the notation, the fact that collections of lines and areas in the drawn diagram can be regarded in various different ways as the reasoning progresses. In the first proposition of the *Elements*, for example, lines that are at one stage in the reasoning regarded as radii of a circle, as they must be to determine that they are equal in length, are later regarded as sides of a triangle, as they must be if we are to conclude that we have constructed the desired equilateral triangle. The diagram has, then, three clearly discernable levels of articulation. First, there are the primitive parts out of which everything is composed: points, lines, angles, and areas. Then there are the geometrical figures that are composed of those primitive parts and form the subject matter of geometry: circles with their centers, circumferences, radii, and areas; triangles with their sides, angles, and areas; squares with their sides, angles, and areas; and so on. And finally, there is the whole diagram, the whole collection of lines, points, angles, and areas, whose various proper parts can be seen now this way and now that. It is precisely because the figures of interest—those at the second, middle level—both *have* (primitive) parts and *are* parts of the diagram as a whole that one can, for instance, introduce circles and radii into a diagram and then take out of it an equilateral triangle.

Geometrical concepts, at least those of concern to Euclid, are given by parts in spatial relation: to be a circle is to be a plane figure all points on the circumference of which are equidistant to a center, to be an equilateral triangle is to be a triangle (itself defined by parts in relation) all sides of which are equal in length, and so on. And because they are, the contents of such concepts can be exhibited in paper-and-pencil drawings. A Euclidean diagram of, for instance, a circle, looks like a circle not so much because it is an instance of a circle as because the relation of its parts displays what it is to be a circle, namely, to have all points on the circumference equidistant to a center. A drawn circle in a Euclidean diagram is, in other words, a Peircean icon; it exhibits the relevant relations of parts, and in so doing generates the appearance of a circle. And because, as I have indicated, a Euclidean diagram is composed of primitive parts that are combined in wholes that are geometrical figures, and are themselves proper

1 In "Boole's Logical Calculus and the Concept-script" written shortly after the 1879 logic, Frege singles out his definition of following in a sequence in particular as being fruitful in this sense (Frege 1880/1881, p. 34).

2 Tappenden (1995), one of the few sustained discussions of our topic, suggests that "Frege's question of how analytic judgments can extend knowledge in the way important mathematical innovations do . . . [is] a question of staggering difficulty" (p. 450). Needless to say, I do not think that it is. Although not easy, the question is (as I hope to demonstrate) tractable if we read Frege's notation, as I do.

parts of the diagram, one can perceptually reconfigure various parts in new ways in order to reach the desired conclusion. The demonstration is fruitful, a real extension of our knowledge for just this reason: because we are able perceptually to take parts of one whole and combine them with parts of another whole to form a new whole, we are able to discover something about our geometrical concepts that was simply not there, even implicitly, in the materials with which we began.

In the seventeenth century, Descartes inaugurated a radically new form of mathematical practice, the practice of algebraic problem solving in the formula language of arithmetic and algebra (see Macbeth 2004). At first this practice seemed utterly different from the paper and pencil reasoning of Euclid; it seemed the work of the pure intellect. (See Descartes' account of mathematical practice in the fifth meditation of his *Meditations on First Philosophy*.) But as Kant would later point out, even this form of mathematical practice is constructive and in a broad sense diagrammatic. One cannot do mathematics as Descartes does it without the essentially written language of arithmetic and algebra; one must see, or imagine seeing, the equations, and must perform, or imagine performing, the needed manipulations on them. Descartes' mathematical practice, like ancient mathematical practice, involves the construction of a kind of diagram, though it is one that is symbolic rather than ostensive. In both cases content is formulated in a system of written marks in a way that enables the discovery of new mathematical truths.

The mathematical practice of constructive algebraic problem solving that was inaugurated by Descartes in the seventeenth century was followed in the nineteenth by what has been called the *Denken in Begriffen* tradition of proving theorems by reasoning deductively from defined concepts (see Laugwitz 1999). Instead of trying to construct solutions to problems in the symbolic language of arithmetic and algebra, mathematicians such as Riemann sought to describe the essential properties of the desired functions and to infer deductively what must be true of a function so described. Mathematical functions, hitherto understood as particular sorts of analytical expressions, as symbolic formulae, were now to be conceived as mappings determined by properties such as continuity and differentiability. For these nineteenth century mathematicians, "the objects of mathematics were no longer formulae but not yet sets. They were concepts" (Laugwitz 1999, p. 305). The task (in Dedekind's words) was "to draw the demonstrations, no longer from calculations, but directly from the characteristic fundamental concepts, and to construct the theory in such a way that it will ... be in a position to predict the results of the calculation".[3]

At first there was no means whereby to set out such thinking visually. And because there was not, many nineteenth century mathematicians held that this new mathematical practice was "too philosophical", that it was not "real mathematics" at all.[4] It would have been natural, then, for a mathematician of the Riemann school to try to develop

3 Quoted in Stein (1988, p. 241).

4 Kronecker, for instance, wrote to Cantor in August of 1884: "I acknowledge true scientific value—in the field of mathematics—only in concrete mathematical truth, or to state it more pointedly, 'only in mathematical formulas'" (quoted in Laugwitz 1999, p. 327). Weierstrass voiced essentially the same sentiment in a lecture delivered in 1886: "even though it may be interesting and useful to find properties of the function without paying attention to its representation ... the *ultimate* aim is always the representation of a function"—that is, an equation, an analytical expression of it (quoted in Laugwitz 1999, p. 329).

a new sort of symbolism, a system of written marks that would do for this new form of mathematical practice what diagrams had done for ancient Greek geometry and the symbolic language of arithmetic and algebra had done for early modern algebra and analysis, a system of written marks that would enable one to set out on paper this new practice of thinking in concepts, *Denken in Begriffen*. Although Kant had thought it impossible, perhaps there *could* be a kind of concept-writing or concept-script, a *Begriffsschrift*. In 1879, Gottlob Frege, a Jena mathematician of the Riemann school (see Tappenden 2006), published a little monograph introducing the written language that he had devised for this purpose. Modeled on the formula language of arithmetic, Frege's *Begriffsschrift* was to enable one to exhibit the contents of mathematical concepts in written marks in a way enabling one to reason deductively from those contents.[5]

2 A new sort of written language

Frege's *Begriffsschrift* was to do for the nineteenth century mathematical practice of reasoning from concepts, *Denken in Begriffen*, what Euclid's diagrams, Arabic numeration, and Descartes' symbolic language had done for earlier forms of mathematical practice; it was to provide a system of written marks within which to reason in mathematics. Frege's thought was "to supplement the formula language of arithmetic with symbols for the logical relations in order to produce ... a conceptual notation" (Frege 1882a, p. 89). "I wish to blend together the few symbols which I introduce and the symbols already available in mathematics to form a single formula language" (Frege 1882b, p. 93). Much as a Euclidean diagram formulates the contents of geometrical concepts in a mathematical tractable way, that is, in a way that enables one to discover new truths in geometry, so Frege's task was to formulate the contents of mathematical concepts generally in a way enabling the discovery of new truths. As one builds numbers out of numbers in the formula language of arithmetic so Frege would build concepts out of concepts in his formula language of pure thought.[6] In *Begriffsschrift* "we use old concepts to construct new ones ... by means of the signs for generality, negation, and the conditional" (Frege 1880/1881, p. 34).

Consider, for example, the concept PRIME NUMBER. To say that a number is prime is to say that it is not divisible without remainder by another number. That is, a number is prime iff it is *not* arithmetically related in a particular way to *any* other numbers. So, if he is to construct such a concept in his language, Frege needs special signs enabling the expression both of negation and of generality. Because the form of mathematical reasoning he is concerned with is deduction, that is, the relation of ground and consequent, Frege furthermore chooses the conditional as his fundamental logical connective (Frege 1880/1881, p. 37). His three primitive signs for expressing logical content as it figures in mathematical concepts such as that of being prime are the

[5] As he notes in the Preface to *Begriffsschrift*, Frege's immediate motivation for devising his concept-script was the difficulty of reasoning rigorously from complex mathematical concepts in written natural language.

[6] See Frege (1880/1881, especially p. 13).

negation and conditional strokes, and the concavity for the expression of higher-level concepts and relations involving generality.

Isolating the primitive logical notions that are needed to serve as the "logical cement" binding old concepts into new, and devising written signs for them, was the easy part. The hard part was to figure out how it is possible to exhibit the contents of concepts *at all*. Frege needed not merely to *say* what a particular content amounts to (as we did above for the case of the concept PRIME NUMBER) but to *exhibit* the inferentially articulated contents of concepts themselves. Much as a drawn circle in a Euclidean diagram iconically displays the content of the concept CIRCLE—that all points on the circumference are equidistant from a center, so that one can infer from the drawn circle that all its radii are equal in length—so Frege needed a way of writing that would display the contents of concepts such as that of being prime in a way that would support inferences. The task was not merely to *record* necessary and sufficient conditions for the application of a concept, what is the case if the concept applies; it was to *show*, to set out in written marks, the contents of concepts as these contents matter to inference. And to do that Frege needed to invent a radically new sort of written language, not merely a new system of written marks but a fundamentally new *kind* of system of written marks.

A drawing of a geometrical figure in Euclid displays the content of the concept of that figure, what it is to be, say, a circle or triangle conceived as a relation of parts. An equation in the symbolic language of Descartes (that is, the language of elementary algebra) does not in the same way display conceptual content. Instead it exhibits arithmetical relations, for instance, that which holds among the lengths of the hypotenuse and other two sides of a right triangle: $a^2 + b^2 = c^2$, where c is the length of the hypotenuse and a and b the lengths of the other two sides. Frege, we will see, effectively combines these two ideas. As Euclid does, he will exhibit the (in this case, inferentially articulated) contents of mathematical concepts, and he will do this in the manner of Descartes, by displaying the (now logical rather than arithmetical) relations that obtain among the constituents of those concepts. In order for this to work, however, Frege must learn to read the symbolic language of arithmetic in a radically new way, in effect, as like a Euclidean diagram whose parts can be conceived now one way and now another.

We begin with a mathematical language, that is, a system of written marks within which to do mathematics, specifically, the formula language of arithmetic. In this system the various signs—the numerals, the signs for arithmetical operations, and so on—all have their usual meanings. In virtue of those meanings, equations in the language serve to display various arithmetical relations that obtain among numbers (or magnitudes more generally). The equation '$2^4 = 16$', for instance, displays an arithmetical relation that obtains among the numbers two, four, and sixteen.[7] In this equation, the Arabic numeral '2' stands for the number two, the numeral '4' stands for four, the numeral '16' stands for sixteen, and the manner of their combination shows the arithmetical relation they stand in. Now we learn to read the language differently, as a fundamentally different kind of language from that it was developed

7 The example is Frege's (1880/1881, pp. 16–17).

to be. Instead of taking the primitive signs of the language to designate prior to and independent of any context of use, as we needed to do to devise the language in the first place, now we take those same signs only to express a sense prior to and independent of any context of use. Only in the context of a whole judgment and relative to some one function/argument analysis will we arrive at sub-sentential expressions, whether simple or complex, that designate something. If, for instance, we take the numeral '2' to mark the argument place, the remaining expression designates the concept FOURTH ROOT OF SIXTEEN. If instead the numeral '4' is regarded as marking the argument place, the remainder designates the concept LOGARITHM OF SIXTEEN TO THE BASE TWO. And other analyses are possible as well. The language is, then, symbolic much as the language of elementary algebra is, but its primitive signs nonetheless function in the way the written marks for points, lines, angles, and areas function in Euclid. In the language as Frege conceives it, the primitive signs only express a sense independent of a context of use. Only in the context of a proposition and relative to an analysis into function and argument do the sub-sentential expressions of the language, whether simple or complex, serve to designate anything.[8]

In both natural language and standard logic—for instance, the logic of Aristotle, of Leibniz, of Boole, and of our textbooks—one begins with concepts. Judgments are then formed by putting the given concepts together. What we have just seen is that Frege's logic is different insofar as "I come by the parts of the thought by analyzing the thought" (Frege 1919, p. 253). "Instead of putting a judgment together out of an individual as subject and an already previously formed concept as predicate, we do the opposite and arrive at a concept by splitting up the content of possible judgment" (Frege 1880/1881, p. 17). "I start out from judgments and their contents, and not from concepts ... I only allow the formation of concepts to proceed from judgments" (Frege 1880/1881, p. 16). By learning to read the symbolic language of arithmetic in a radically new way, its primitive signs as only expressing a sense independent of the context of a proposition and relative to an analysis, Frege is able to exhibit the contents of concepts in his new language, to display in a written array of marks those contents as they matter to inference.

It is obvious that natural language does not function in the way that Frege suggests. Nor is the way Frege reads the formula language of arithmetic the way that language was originally designed to work, or even how it *could* have been originally designed to work. In order to learn to read a system of written marks the way Frege reads the formula language of arithmetic, one must *first* know how to read it as it was designed to be read, each primitive sign as having its meaning, its significance or designation, independent of a context of use. The new form of language Frege introduces is an essentially late fruit of intellectual inquiry; it is only after one has learned to read the language the usual way that one can learn to read it as Frege intends. We need first to know what, independent of any context of use, Frege's conditional and negation strokes mean (that is, their usual truth-functional meanings), and what second-level concept his concavity designates (namely, the concept, as we can put it, UNIVERSALLY

8 This feature of Frege's notation is explored at length in Macbeth (2005).

APPLICABLE[9]), before we can learn to read *Begriffsschrift* formulae in Frege's new way, as presenting Fregean thoughts that can be analyzed into function and argument in various ways.

In order to design a *Begriffsschrift* for reasoning from concepts in mathematics Frege needed to find a mathematically tractable way of exhibiting the inferentially articulated contents of mathematical concepts. And, we have just seen, he solved that problem by learning to read an already meaningful language in a radically new way. Instead of beginning with primitive signs for concepts, he would arrive at an expression designating a concept only through a function/argument analysis of a whole Fregean thought. A concept word so generated can be highly articulated; that is, it can be composed of many primitive signs of the language in some particular array. It is in just this way that one exhibits, fully displays, the inferentially articulated content of a concept in Frege's system. The inferentially articulated content that is expressed by a complex concept word in Frege's language, which is itself directly a function of the senses of the primitive signs used in its expression, is a Fregean sense, one that contains a mode of presentation of the relevant concept.

We have the language we need, a language within which to exhibit the inferentially articulated contents of concepts. Now we need to think about the definitions that provide the starting point for proofs in the mathematical practice of concern to Frege and about the rules governing legitimate moves from one *Begriffsschrift* formula to another in such proofs.

3 Definitions and rules of inference in *Begriffsschrift*

In Euclidean diagrammatic practice, definitions do not provide a starting point for proofs; definitions in that system belong to the antechamber, the preamble or preparation that one gives antecedent to the actual work of mathematics. In Euclid, it is not the definition but the diagram that formulates conceptual contents in a way that enables demonstration. In the mathematical practice that emerged over the course of the nineteenth century, the mathematical practice of concern to Frege, definition *is* the starting point for proof. A definition is, in this case, "a constituent of the system of a science" (Frege 1906, p. 302). And what it does, Frege tells us, is to stipulate that a newly introduced simple (that is, unanalyzable) sign is to have precisely the same meaning or designation as some complex expression formed, in the first instance, out of primitive signs. The definition exhibits, in the *definiens*, the inferentially articulated content of some concept, and introduces, in the *definiendum*, a simple sign that has, by stipulation, precisely that same meaning. The newly introduced simple sign designates just the concept that is designated in the definiens. But it is not, contrary to what Frege says, merely an abbreviation, a shorthand form of what is given in the definiens.[10]

[9] Although Frege's concavity is widely read as a universal quantifier, it is not one. It is a sign for a second-level concept. Thus, for instance, a sign formed from Frege's concavity together with the conditional stroke can be read as a sign for the second-level relation of subordination. See Macbeth (2005, §3.3).

[10] See, for instance, Frege (1879, §24), also Frege (1914, pp. 208–209).

Because defined signs are simple and unanalyzable, to prove a theorem the expression of which involves defined signs is to prove something about the *particular* concepts that are designated by those signs. The same is not true if all defined signs are replaced with their definitions. As we will see in some detail below, if one is to prove something about the particular concepts of interest then one needs *both* the defined simple signs designating those concepts and their definitions; one needs the simple, unanalyzable signs to ensure that one's theorem is about the particular concepts designated by those signs, and one needs their definitions if logical relations are to be discovered to obtain among those very concepts. Definitions are in this way and for this reason "a condition for insight into the logical linkages of truth" (Frege 1914, p. 302).

In Euclidean diagrammatic reasoning, the mathematician's task is to find the diagram that will provide the medium of reasoning from one's starting point, a given line or triangle, say, to the desired endpoint, for instance, an equilateral triangle drawn on the given straight line, or the equality of the sum of the angles to two right angles. In reasoning in Frege, the mathematician's task is instead to find a path, that is, a sequence of inferences, from the given definitions to the desired theorem. Somehow, the defined signs that originally occur in different definitions must be joined together in the derived theorem. As we might think of it, our paper-and-pencil task is to *construct* (in something like Kant's sense) the theorem on the basis of the given definitions. Obviously, then, we need some rules of construction, rules like those found in algebra governing what one is allowed to write given what one has already written. And here, because my philosophical purposes are different from Frege's mathematical ones, I want to diverge from Frege's practice, though not from the spirit of that practice.

Frege's aim in his proof of theorem 133 in Part III of the 1879 logic is to take the first step in his logicist program of showing that arithmetic is merely derived logic. Ultimately, what Frege wants to show is that all the concepts of arithmetic can be defined by appeal only to logical concepts and that all laws of arithmetic can be derived on the basis of purely logical laws. What matters to him, then, is that his proofs be maximally rigorous, which means in turn that, "because modes of inference must be expressed verbally", that is, in natural language rather than in *Begriffsschrift*, only one mode of inference is to be employed in the 1879 logic (Frege 1880/1881, p. 37). All other modes of inference are instead given as formulae in Part II of *Begriffsschrift*, either as axioms or as theorems derived from those axioms according to Frege's one mode of inference. In Frege's presentation, then, both definitions and modes of inference take the form of formulae, and this can easily lead one to think that they play essentially the same role in a *Begriffsschrift* proof. But they do not. The starting point for a proof in *Begriffsschrift* is not all those various formulae but only the definitions. The proof is, as Frege says, "*from* the definitions ... *by means of* my primitive laws".[11] Thus, if we are not concerned with the mathematical project of logicism but want instead philosophical understanding of just how

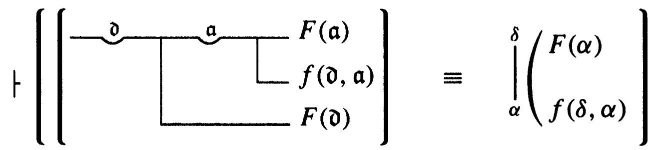
[11] Frege (1880/1881, p. 38); emphasis added.

reasoning from definitions works in Frege's system, it is better explicitly to treat the primitive and derived modes of inference introduced as formulae in Part II of the 1879 logic instead as rules licensing inferences, that is, as rules governing what one is allowed to write given what one has already written beginning with the definitions.

In Part III of *Begriffsschrift*, Frege provides four definitions and on the basis of those definitions proves an important theorem (numbered 133) relating three of the four defined concepts.[12] In the course of the proof, defined signs that originally appear in different definitions are brought together, joined, in a single formula that displays the logical relation that is proven to obtain among the concepts designated by those defined signs. As we will see, two different sorts of inference are involved in this process of proof. First, there are those inferences, which I will call *linear* inferences, that serve merely to alter a given formula in some way, say, by reordering its conditions or by adding a new condition. The second form of inference, a *joining* inference, essentially involves two or more different formulae, both ultimately derived from a definition (or definitions), and serves to combine content from those different formulae using, in effect, some version of hypothetical syllogism.[13] Beginning with the definitions, the strategy of the proof is, first, to alter in various ways the formulae initially given as definitions so that they share content, content that occurs once in the consequent of a conditional and once as the antecedent. Given this shared content, the remaining content, originally derived from two different definitions, can then be joined into a single judgment. The process is then repeated until all the defined signs are appropriately joined in the desired theorem. Figure 1 provides an overview of all the joins in Frege's proof of theorem 133.[14]

4 Some details

Part III of *Begriffsschrift* is, Frege tells us (§23), "meant to give a general idea of how to handle" his system of written signs. It is to show, by example, how to prove theorems deductively on-the basis of explicitly formulated definitions in mathematics. We begin with four definitions, that of being hereditary in a sequence:



[12] Frege describes theorem 133 as "a sentence which, it seems to me, is indispensible in arithmetic, although it is one that commands little attention, being regarded as self-evident" (Frege 1880/1881, p. 38).

[13] These two forms of inference correspond to the two sorts of moves one can make in algebraic reasoning. Linear inferences correspond to moves governed by the familiar rewrite rules of the system. Joining inferences correspond to moves that in algebra are governed by the rule that one can put equals for equals. Much as hypothetical syllogism does, this second sort of move enables one to combine content from two different formulae in algebra. See Macbeth (forthcoming) for further discussion.

[14] In the proof as Frege presents it, there are some apparent divergences from this overall strategy; it is not hard to show that in fact the course of the proof is as described.

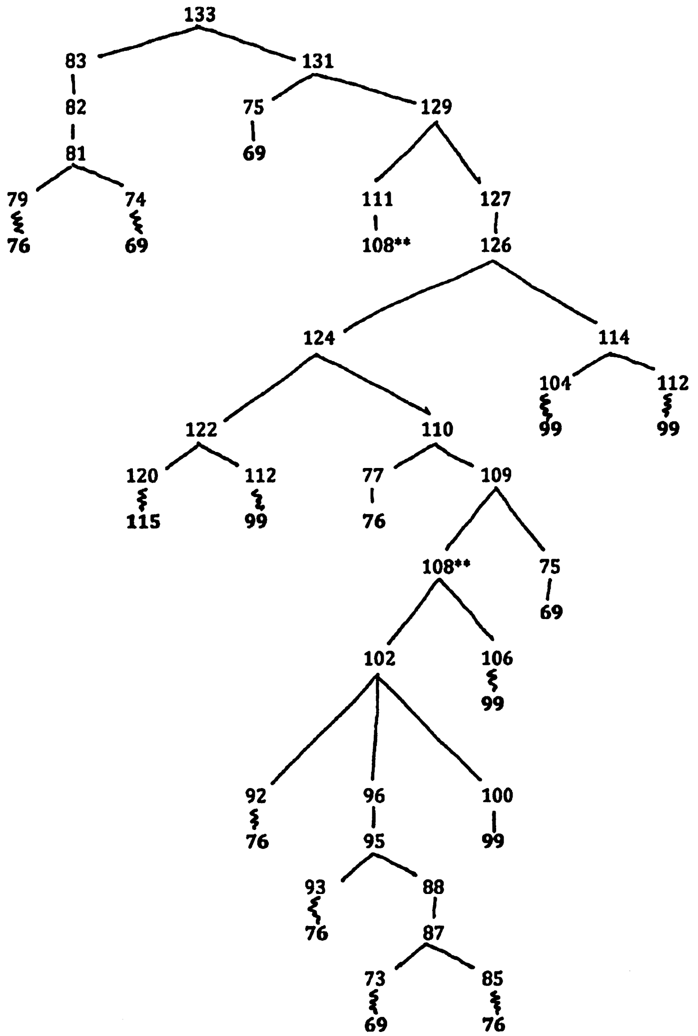
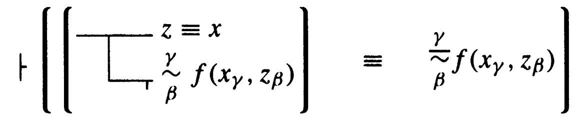


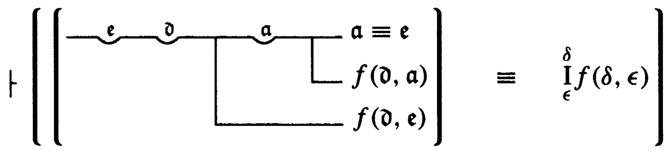
Fig. 1 Outline of the derivation of theorem 133 in part III of the *Begriffsschrift*. *Straight lines* indicate inferential steps, either linear inferences or joins; *wavy lines* indicate that a series of linear inferences have been suppressed

that of following in a sequence, which Frege singles out as being a paradigm of a fruitful definition in his sense (Frege 1880/1881, p. 34):

that of belonging to a sequence, which is merely truth-functional—z belongs to the f-sequence beginning with x just in case either z follows x in the f-sequence or z is identical to x—and hence not fruitful in Frege's sense:

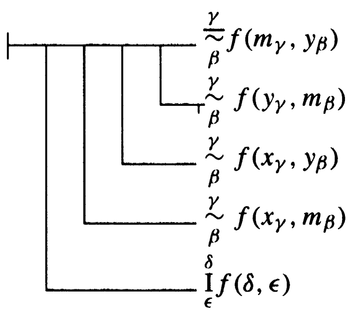


and finally, that of being a single-valued function, which is a necessary condition of the truth of the theorem Frege aims to derive:



Each of these definitions exhibits the inferentially articulated contents of the relevant defined concepts. They provide in the *definiens* (to the left of the triple-bar) a *picture* or *map*, that is, a Peircean icon, of the sense of a concept word, and in the *definiendum* (to the right of the triple-bar) a simple sign, newly introduced, that is stipulated to have the same meaning as the complex of signs on the definiens. And as we have seen, the contents of concepts *can* be exhibited in this way in Frege's notation in virtue of the fact that independent of an analysis, a particular way of regarding it, a *Begriffsschrift* formula only expresses a sense, a Fregean Thought. In a definition, the *definiens* is a concept word that, on the analysis that is stipulated by the definition, exhibits the sense of the relevant concept word. In the judgment that immediately follows from a definition, as in any judgment in Frege, this sense can be regarded in various ways. Because such concept words both have (primitive) parts and are themselves parts of whole formulae, they can, like the parts of a Euclidean diagram, be regarded now this way and now that. The *definiendum* is a concept word for that same concept, but unlike the *definiens* it is a simple sign. Because it is simple, it cannot in the context of a judgment be variously analyzed.

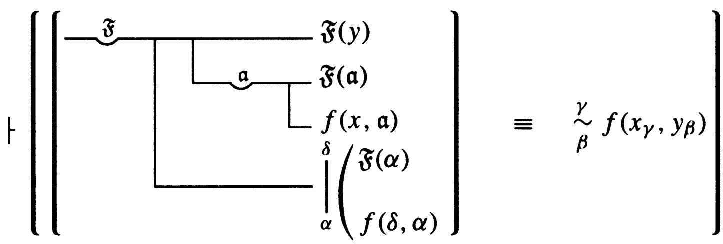
What we want to derive is theorem 133 according to which if f is a single-valued function and y and m follow x in the f-sequence, then either m follows y or y belongs to the f-sequence beginning with m. That is, as we can also say in light of the definition of belonging to a sequence, if f is single-valued and y and m follow x in the f-sequence, then either m follows y or y follows m or m and y are identical. In *Begriffsschrift*:



The task of the derivation is to join the various defined signs that originally occur in three different definitions in the theorem in the way shown. And as already indicated, there are two parts to the process of proof, the preparation and the joining of contents suitably prepared.

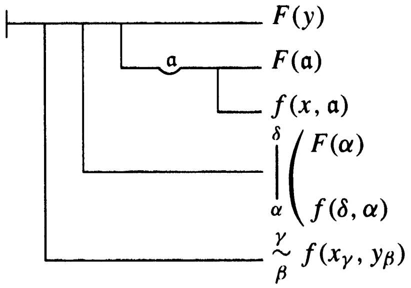
The starting point is two definitions, '||—definition-of-α ≡ α ||' and '||—definition-of-β ≡ β ||', where what is to the left, the *definiens*, is a complex expression formed from primitive signs, and perhaps also previously defined signs, and what is on the right, the *definiendum*, is a simple sign newly introduced that is stipulated to have the same meaning as the *definiens*. The first step in the preparation is to transform both identities into conditionals, into, for instance, 'α-on-condition-that-[definition-of-α]' and '[definition-of-β]-on-condition-that-β'. That is, in one the *definiens* is made the condition and in the other it is the *definiendum* that is made the condition. (In practice, other combinations can also occur.) Then one transforms the two conditionals in various ways according to rewrite rules until they share content. That is, we derive something like 'α-on-condition-that-[definition-of-α]*', and also something of the form '[definition-of-β]*-on-condition-that-β', where '[definition-of-α]*' is identical to '[definition-of-β]*'. (In practice, things are of course not this simple.) Then we use some form of hypothetical syllogism to join the defined signs 'α' and 'β' in a single judgment as mediated by the common content: 'α-on-condition-that-β'. The process is then repeated until all the defined signs occurring in theorem 133 are appropriately joined.

Consider, for example, the chain of linear inferences that effect various transformations of definition 76 preparatory to the join with formula 74 (itself derived from definition 69 by a series of linear inferences) that yields theorem 81.[15] The definition, again, is this:



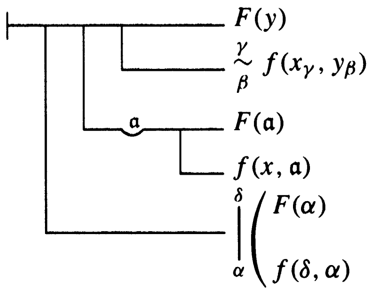
The first step is to convert this definition, more exactly the judgment that immediately derives from it, into a conditional judgment with the outer-most concavity removed, theorem 77:

15 See the top left part of the outline of the proof in Fig. 1.



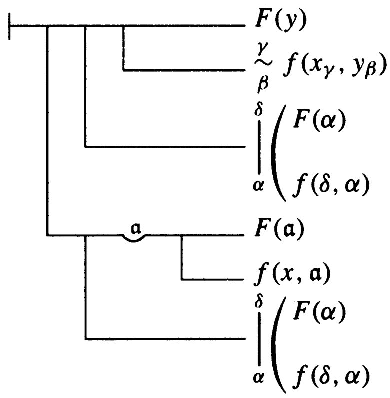
That is, we make the defined sign, the *definiendum*, a condition on the content that is the *definiens* of the original definition.

Now we make various modifications to this formula in a series of linear inferences preparatory to a joining inference. First, we switch around the three conditions, licensed by one of Frege's many reordering theorems, to derive theorem 78:



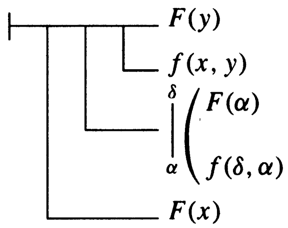
Notice that in order to do this we had to regard theorem 77, derived from the definition, in a new way. In order to see that theorem 77 is a conditional formed from definition 76, we had to regard the lowest condition on it as the condition on the rest of the formula conceived as the conditioned content, the consequent. But in order to see that we can reorder the conditions as we just did to get theorem 78, we needed to treat everything except —$F(y)$ as a condition on that content as the conditioned content. And the point applies generally to reasoning in *Begriffsschrift*. Conditions that are at one point in the reasoning regarded as parts of the conditioned judgment *in* a conditional are at a different stage in one's reasoning regarded instead as conditions *on* a conditioned judgment. Just as is the case in reasoning in Euclid, the perceptual skill of seeing now this way and now that is required if one is to follow the course of the reasoning.

Now we reorganize the content as licensed by Frege's second axiom of logic to yield theorem 79:



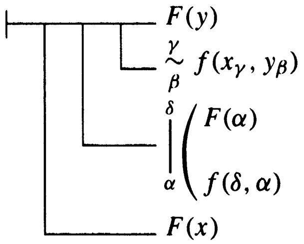
That is, we remove the lowest condition in theorem 78 and reattach it to both the condition and the conditioned content of the remainder suitably regarded. The preparation is complete.

We assume a similar preparation for the second formula needed in the join, namely, theorem 74 derived by a series of linear inferences from the definition of being hereditary in a sequence:



Notice now that the lowest condition (itself a conditional) in theorem 79 is identical—save for the presence of the concavity in theorem 79—to the conditioned judgment in theorem 74, regarded as having $—F(x)$ as the sole condition on that conditioned content. But we know (from *Begriffsschrift* §11) that we can insert a concavity in the relevant place in theorem 74, and hence, together, these two formulae yield, by hypothetical syllogism, theorem 81[16]:

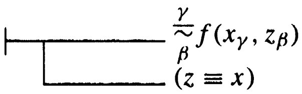
[16] In his actual practice Frege does not directly join content from two formulae as licensed by some version of hypothetical syllogism. Instead he forms, from one of the two original formulae, a new formula that together with the other of the two original formulae yields the desired conclusion by modus ponens. Theorems 5, 7, 19, and 20, for instance, are all used this way. But it is easy to see that all can equally well be treated as rules governing two premise inferences directly from the original two formulae, as we have done here.



In this formula the lowest condition, $—F(x)$, derives from theorem 74, ultimately from the definition of being hereditary in a sequence, and the rest of the formula derives from theorem 79, ultimately from the definition of following in a sequence. Much as in reasoning through a diagram in Euclid one *perceptually* joins parts of different wholes to form new wholes, so here we *literally* join, by means of hypothetical syllogism, parts of different wholes to form a new whole, ultimately, the whole that is theorem 133.

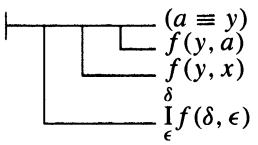
Now we focus on the series of joins at the heart of the proof.

112.



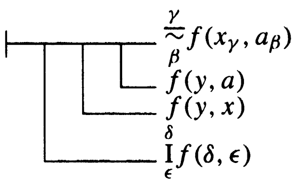
This formula derives directly from the definition of belonging to a sequence by a series of linear inferences governed by the rewrite rules of Frege's system.

120.



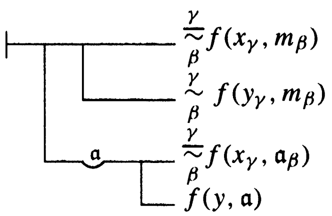
This formula is derived by a series of linear inferences from the definition of being a single-valued function.

122.



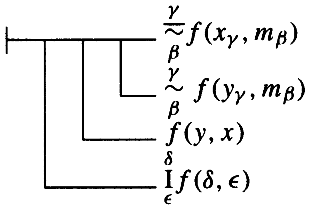
Because the condition in 112, with a for z, is identical to the judgment, on three conditions, in 120, the two can be joined, by a variant of hypothetical syllogism, in theorem 122.

110.



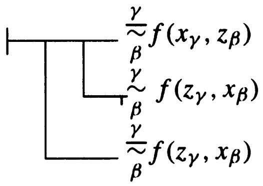
Theorem 110 is a nontrivial result of a chain of linear and joining inferences beginning with definitions 69, 76, and 99. Notice that its lowest (conditional) condition is identical, save for the presence of the concavity, to the (conditional) conditioned judgment in 122.

124.



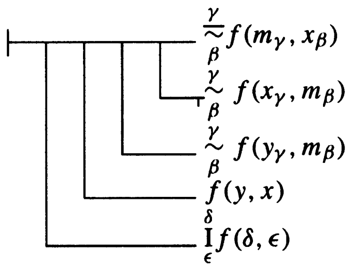
Theorem 124 is the result of a join of 122 and 110. The bottom two conditions derive from 122, the rest from 110.

114.



Theorem 114 is derived from a join of two judgments both of which derive ultimately from the definition of belonging to a sequence. Notice that the lowest condition in 114, with x for z and m for x, is identical to the topmost conditioned content in 124.

126.



Derived from a join of 124 and 114, with x for z and m for x in 114, theorem 126 is very close to what we want: it looks almost like theorem 133. (Nevertheless, there is nontrivial work yet to be done before the construction is completed.)

Figure 2 provides a more graphic display of these joins. The rest of the proof is essentially similar.

5 A real extension of our knowledge?

Frege claims that his proof of theorem 133 extends our knowledge, that it is ampliative in some way that, say, the derivations in Part II of *Begriffsschrift* are not. But what, aside from complexity, does the proof of 133 have that the derivations in Part II do not? Although I have not tried to show it here, both require us to regard a formula now this way and now that. Both involve the construction of various formulae to take us from something we have to something we want. Both require a kind of experimentation to determine not only what rule to apply but, in cases that involve the addition of a condition or conditioned judgment, what it is useful to add. Both seem, in other words, to be theorematic rather than corollarial in Peirce's sense (see Shin 1997). Finally, although most of the derivations in Part II do not, some of those derivations do involve inferences that join content from two axioms just as Frege's proof of 133

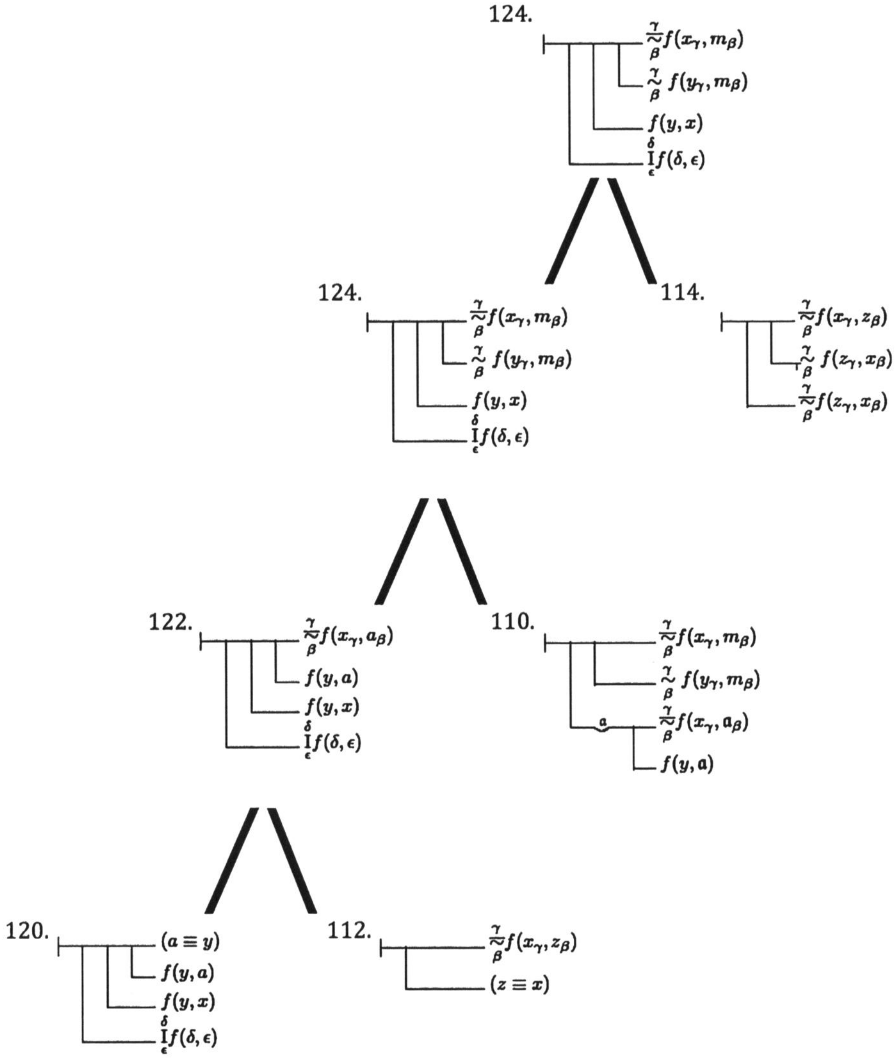


Fig. 2 Some central joins in Frege's proof of theorem 133

involves inferences that join content from two definitions.[17] Why should we think that a theorem that joins content from two definitions extends our knowledge, though a theorem that joins content from two axioms does not?

The beginnings of an answer can be found in the fact that definitions and axioms of logic, as Frege understands them, are essentially different. An axiom of logic is a judgment, a truth. It is (or at least should be) immediately evident (*einleuchtend*), but

[17] For example, in Frege's derivation of theorem 33 from axioms 31 and 28, he constructs theorem 32, on the basis of axiom 31, to govern the passage from 28 to 33. But it is clear that we can instead take theorem 7 (used in that construction) directly to license a two-premise inference from 31 and 28 to 33, one that joins content from the two axioms.

it is nonetheless a substantive truth that does not go without saying. A definition, we have seen, is not a judgment; it is a stipulation. Such a stipulation does immediately yield a judgment, but this judgment is one that is, in light of the stipulation, utterly trivial. It is not merely immediately evident as an axiom is (or should be); it is self-evident (*selbstverständlich*). It goes without saying. But as we have also seen, although they are trivial in themselves, the judgments that derive from definitions enable one to discover, by way of proof, logical bonds among the concepts designated by the defined signs. In the case of a demonstration on the basis of definitions of concepts, we are not merely joining content in a thought that can be variously analyzed, as is the case in the derivation a theorem of logic from axioms; we are discovering logical bonds among the *particular* concepts that are designated by the defined signs. If in Frege's proof of theorem 133 we were to replace all the defined signs by their definitions, then we *would* have a mere theorem of logic, one subject to many, many different analyses. The theorem would not be about the concepts of following in a sequence, belonging to a sequence, and being a single-valued function at all. It would not be a theorem in the theory of sequences. It is only if we have definitions of concepts that we can forge logical bonds *among those very concepts*. And only in that case can the construction extend our knowledge of those concepts by revealing their logical relations one to another.

In the course of a proof such as that of theorem 133, the simple defined signs are needed if what is to be established is to be unequivocally about the concepts of interest. But their definitions are needed if anything about those concepts is to be established. Here, then, we have something new that is made possible *only* in light of definitions, that is, judgments that involve, on the one hand, a simple sign, the *definiendum*, and on the other, a complex sign, the *definiens*, that exhibits the inferentially articulated content of the relevant concept. In this case, and only in this case, content that is derived ultimately from the *definiens* of two different definitions, if it can be brought to be identical in the two cases, can be used to reveal a logical bond between the concepts designated by the defined signs. That it is these concepts and no others that are joined is determined by the defined signs; that they *can* be joined is made possible by the fact that the contents of those signs are also given in various complex expressions, expressions that are variously analyzable.

In Part II of the 1879 logic Frege derives various theorems he will need in his proof, in Part III, of theorem 133. These theorems, and the axioms from which they are derived, function in the latter proof as rules licensing the linear and joining inferences that take us from Frege's definitions to his theorem. As Frege himself notes, his axioms and derived theorems in Part II were chosen precisely because and insofar as they are needed in the proof of theorem 133 of Part III: "Apart from a few formulae introduced to cater for Aristotelian modes of inference, I only assumed such as appeared necessary for the proof in question", that is, the proof of theorem 133 (Frege 1880/1881, p. 38). Although the axioms and theorems of Part II are truths on Frege's view, they have, in other words, little intrinsic interest; they are valuable not in themselves but instead for what they enable one to prove on the basis of defined concepts. Independent of the proof of theorem 133, one would have no reason to derive this rather than that theorem of logic, no reason to start, or to stop, with any particular axioms or theorems of logic. The interest of the "sentences of pure logic" that we find in Part II of Frege's *Begriffsschrift* lies in "the fact that they were adequate for the task" that is undertaken in Part III (Frege 1880/1881, p. 38).

I have suggested that Frege's proof of theorem 133 is ampliative because it reveals logical relations among concepts and does so by combining, in joining inferences, parts of different wholes into new wholes. Much as reasoning through a diagram in Euclid does, such a course of reasoning realizes something new that had the potential to be derived but was in no way implicit in the starting point of the derivation. Perhaps it will be objected that we can also formulate definitions in standard logical notation, and indeed, can reproduce the whole of Frege's proof in standard notation. The deduction written in standard notation is not ampliative. So why think that the deduction written in Frege's notation is ampliative? Furthermore, when Frege discusses ampliative proof in the long Boole essay and in *Grundlagen*, he focuses in particular on proofs from definitions that do not merely list characteristics but instead draw new lines (as he thinks of it). If Frege's definitions can draw new lines then so can definitions formulated in standard notation. But again, no deduction in standard notation, whether from "fruitful" definitions (those that draw new lines) or from unfruitful ones, can extend our knowledge in any real sense. How can it possibly be that a proof written in Frege's concept-script is ampliative when that very same proof written in standard notation is not? Either both are ampliative or neither is. Since the proof in standard notation is not ampliative, it must be the case that Frege's *Begriffsschrift* proof likewise is not ampliative.

This little argument, although understandable, is wrong. In fact one cannot formulate Frege's definitions, in particular, those he thinks of as fruitful, in standard notation. And although one can in a certain sense rewrite Frege's proof in standard notation, something essential is lost in the translation. I will first try to clarify what it is that is lost, and then I will argue that the difference this ingredient makes is just the difference between ampliative and merely explicative proof.

It will help to begin with a very different, but (we will see) fundamentally analogous case, that of Roman and Arabic numerals. Obviously any natural number expressed in Arabic numeration can be "translated" into the system of Roman numeration. If you can express a natural number in the one system you can express it in the other. Both can furthermore be used to record how many things of some sort one has; both tell you, each in its own way, how many. But the system of Arabic numeration also does something more. Although numerals in that system can be used to record how many, the system was in fact not developed for that purpose, which is already adequately served by the older Roman numerals. Arabic numeration was developed as a system of written marks within which to do arithmetical calculations. Unlike a Roman numeral, which only records how many, an Arabic numeral formulates content as it matters to arithmetical operations; it enables one to compute *in* the system of signs. It is just this content that is lost when one moves over to Roman numeration. Roman numeration does not formulate content as it matters to arithmetic. All it does is provide a means of recording how many.

The systems of Roman and Arabic numeration are essentially different insofar as while the Roman system serves only to record, to *say*, how many, the Arabic system serves to formulate arithmetical content, to *show* in a written display content as it matters to arithmetical calculations. Indeed, unlike the Arabic system, the system of

Roman numeration is not essentially written at all. The roles played by the written signs, the 'X' for ten things, 'V' for five, 'I' for one, and so on, could, for instance, be played as easily by colored tokens: a blue token (say) could stand for a single thing, a red one for a collection of five things, a yellow one for a collection of ten, and so on. Putting a red token, two yellow ones, and three blue ones into a little bag would then serve as a record of how many, namely, in this case, 28.[18] Unlike the system of Roman numeration, the Arabic numeration system *is* essentially written. It is a positional numeration system that utilizes the expanse of the page, and thereby the relative locations of the signs, to exhibit arithmetical content. Although one can use Arabic numeration to record how many, that system's primary purpose is the quite different one of providing a written system of signs within which to perform paper-and-pencil arithmetical calculations. This, again, is what is lost in the translation to Roman numeration.

Our standard notations are like the system of Roman numeration insofar as they serve to record truth-conditions, what is the case if a sentence is true, and are not essentially written. One can *speak* sentences of ordinary logic as well as read them, *say* what the relevant truth-conditions are as well as write them. This is furthermore just what one would expect given that our standard logics aim to capture patterns of natural language reasoning. Natural language is first and foremost a spoken (or signed) language and a vehicle of communication. Frege's notation, we have seen, is essentially different. It was designed not to make the logical forms of natural language utterances more perspicuous but to exhibit in a kind of a diagram the inferentially articulated contents of concepts as they matter to mathematical reasoning. Much as Arabic numeration enables one to exhibit the contents of numbers as they matter to arithmetic, and Euclid's diagrams enable one to exhibit the contents of geometrical concepts as they matter to diagrammatic reasoning, so Frege's concept-script enables one to exhibit the contents of concepts as they matter to deductive reasoning in the mathematical practice that developed over the course of the nineteenth century and remains the norm still today. And in order to do that, we have seen, the notation has to function in a very peculiar way: independent of a context of use the primitive signs of Frege's concept-script do not designate but only express a sense. Frege's *Begriffsschrift* was designed in this way to exhibit inferential content, content as it matters to inference. This purpose is not served, we will see, by tracing truth-conditions. Much as translating an Arabic numeral into Roman numeration deprives us of the capacity to do arithmetic in the system of signs, so to translate a formula of Frege's concept-script into standard notation is to deprive us of the capacity to reason from defined concepts in the system of signs.

We tend to think of meaning in terms of truth-conditions: what a sentence means, we think, is given by what is the case if it is true. And we further tend to think that what a valid deduction shows is that if the premises are true then the conclusion is also true. We think, in other words, that the truth-conditions of a sentence give one everything that is necessary for a correct inference, that truth-conditions exhaust inference

[18] Roman numeration as generally used is written insofar as we write (say) four as 'IV' rather than as 'IIII', and here the relative spatial locations of the two signs matter. This is, however, a late refinement of the system; it is no part of the basic conception of the system of Roman numeration.

potential. But as Frege already indicates in "On Sense and Meaning" (Frege 1892), this is wrong, and it is wrong even in the case of natural language reasoning, for example, in reasoning involving ordinary proper names. The *truth-conditions* of, say, the sentences 'Hesperus is a planet' and 'Phosphorus is a planet' are identical: the two sentences are about one and the same object, whether or not anyone knows it, and they obviously ascribe one and the same property to that object. But the *inferential significance* in the two cases is different. It does not *follow* from the fact that one of these sentences is true that the other is true as well, which is why it is possible for a rational person to believe one without also believing, perhaps while positively denying, the other. It is only if one is given *both* that, say, Hesperus is a planet *and* that Hesperus is identical to Phosphorus that one can deduce that Phosphorus is a planet. Although the two names 'Hesperus' and 'Phosphorus' designate one and the same object, they express different senses and as a result have different inference potentials. But if so then we need to distinguish between meaning in the sense of truth-conditions, what *is* the case if a sentence is true, and meaning in the sense of inference potential, what *follows* if a sentence is true. The two notions are different.[19] Indeed, we have seen as much already in the case of *Begriffsschrift* definitions. Although the *definiens* and the *definiendum* designate the same concept, they have essentially different inferential roles: the *definiens* is variously analyzable in the context of a judgment; the *definiendum* is not. It follows directly that they express different senses.

Much as both Arabic numerals and Roman numerals can be used to record how many, though they do so in different ways, so both a sentence of standard notation and the corresponding *Begriffsschrift* formula can be used to record truth-conditions, though they do so in different ways. In standard notation truth-conditions are provided directly. The primitive signs are meaningful independent of any context of use and are put together in a sentence that serves thereby to record what is the case if the sentence is true. (The notation is in this regard quite like the formula language of algebra as it was originally designed to be read.) In Frege's concept-script, the primitive signs do not designate but only express a sense independent of a context of use, and as a result, truth-conditions are available only relative to an analysis into function and argument. The language was not designed to trace truth-conditions, any more than Arabic numeration was designed to record how many. *Begriffsschrift* was designed to exhibit inference potential, and because it was well designed one can in fact reason *in* the language. Standard notation was not designed as a language within which to reason, and it cannot be so used. One can use the language to *record* what else is true if one's given premises are true, but the language does not enable reasoning *in* the system any more than Roman numeration enables arithmetical calculations.

In our logics it is assumed that inference potential is given by truth-conditions. Hence, we think, deduction can be nothing more than a matter of making explicit information that is already contained in one's premises. If the deduction is valid then the information contained in the conclusion must be contained already in the premises; if that information is not contained already in the premises, even if only implicitly, then the argument cannot be valid. Indeed it is true generally that in cases in which

[19] See also Macbeth (2005, Chaps. 3 and 4).

the inference potential of a sentence is exhausted by its truth-conditions, deduction is merely explicative. To say, in such a case, that a proof is deductive just is to say that it is merely explicative. But, we have seen, inference potential is not invariably given by truth-conditions, either in natural language or in Frege's concept-script. In these languages, the two notions are different. And because they are different, the question whether one has a deduction or not is different from the question whether the inference is ampliative or not.

Frege indicates that the inference potential of a mere truth-function, even one that is formulated in his concept-script, is given by what is the case if the truth-function is true. Frege's axioms in Part II, for example, are merely truth-functional. And because they are, all the formulae that are derived from those axioms in Part II of *Begriffsschrift* are contained already implicitly in Frege's axioms. And of course the same is true in standard logic. Furthermore, in standard logic, definitions, even those that involve quantifiers, do nothing to change this. A definition expressed in standard notation records necessary and sufficient conditions for the application of a concept. It does not show what follows if the concept applies but only says what is the case if it applies. A definition in standard notation (even one involving quantifiers) is merely truth-functional. A definition in a language such as Frege's is, or at least can be, essentially different. Although the definition of, for instance, belonging to a sequence is merely truth-functional, merely Boolean, the definition of following in a sequence is not. Because it involves the concavity, it draws new lines. And hence, Frege suggests, what can be inferred from it is contained in it not implicitly but only potentially. We need to understand, then, what is the difference between being contained implicitly and being contained potentially, and why Frege would think that the representation of generality is the difference that makes the difference between the two.

Frege indicates that using his concavity together with his signs for negation and the conditional he is able to exhibit an inference potential that is different from the relevant truth-conditions, not merely what is the case if some concept is applied but also what follows if it is applied. To illustrate what I think he means I want to consider a very simple case, that of the pejorative term 'Boche'. As Dummett points out, "the condition for applying the term to someone is that he is of German nationality; the consequences of its application are that he is barbarous and more prone to cruelty than other Europeans" (Dummett 1972, p. 454). That is, the truth-conditions of 'NN is a German' and 'NN is a Boche' are identical; in both cases, the sentence is true iff the one referred to has the property of being of German nationality. But the inference potentials of the two sentences are very different insofar as it can be inferred from the latter but not the former that NN is barbarous and more prone to cruelty than other Europeans. It follows directly that the content of 'Boche' is not Boolean. 'Boche' does not mean 'German and barbarous'. Nor even does it mean 'German and barbarous-*because*-German' given that, as Dummett indicates, to call someone a Boche is not to *say* that they are barbarous. To call someone a Boche is to *say* only that they are German. And yet more follows from calling someone a Boche than it does from calling someone a German, and it does because to call someone a Boche is to call them a German *and* to endorse an inference license from something's being German to its being barbarous and more prone to cruelty than other Europeans. Something is a Boche just in case it is a German *and* being German entails being barbarous and more prone to cruelty than other Europeans. Thus, as Dummett says, "someone who rejects the word does so because he does not want to permit a transition from the grounds for applying the term to the consequences of doing so" (Dummett 1972, p. 454). Because the concept contains as part of its content this inference license, it can be expressed in Frege's *Begriffsschrift* only by means of the concavity together with the conditional stroke.[20]

According to Dummett to call someone a Boche is not to say that they are barbarous, even implicitly. But an application of the concept to someone nevertheless provides all the resources that are needed to draw the inference to the conclusion that the person in question is barbarous. As we can put the point, the conclusion is contained *potentially* in the original ascription in the sense that the ascription provides everything needed to draw the conclusion. Nevertheless the conclusion is not *already* drawn by the ascription, even if only implicitly. One has not said that the person is barbarous, despite the fact that, again, by saying that the person is a Boche, one has everything needed to infer that he or she is barbarous. This, then, is the difference that Frege's concavity makes: it enables the expression of the contents of concepts in cases in which those contents themselves include inference licenses.

I have suggested that we read Frege's notation diagrammatically, as exhibiting conceptual content as it matters to deductive inference in a way that is analogous to the way a Euclidean diagram formulates content as it matters to reasoning in that system. And much as a demonstration in Euclid enables an extension of our knowledge by revealing something new that is achieved by (perceptually) putting together parts of different wholes into new wholes, so, I have suggested, a proof from definitions in Frege's concept-script enables an extension of our knowledge by revealing something new that is achieved by (literally) putting together in joining inferences parts of different wholes into new wholes. Perhaps it will seem that this cannot be right given that, as Frege himself says, "surely the truth of a theorem cannot really depend on something we do, when it holds quite independently of us" (Frege 1914, p. 207). Indeed, Frege makes the remark in the context of a discussion of the status of Euclid's postulates aimed at correcting the misimpression that postulates are somehow essentially different from axioms. A postulate is not to be seen as a rule governing the actual drawing of lines but instead refers, Frege thinks, to an objective conceptual possibility.[21] "So," Frege concludes, "the only way of regarding the matter is that by drawing a straight line we merely become ourselves aware of what obtains independently of us." In a proof, whether in Euclid or in Frege, the truth that is derived obtains independently of

[20] Notice that the quantificational "translation" of the definition does not give what is wanted: x is a Boche $=_{df} \mathrm{G}x \& (\forall x)(\mathrm{G}x \supset \mathrm{B}x)$. If that *were* the definition, to call someone Boche would be to *assert* not only that the person is a German but also that each and every German is barbarous, and hence implicitly that the person in question is barbarous. In that case, the conclusion is contained already in the premises needing only to be made explicit. See Macbeth (2005, Chap. 1), for further discussion of the difference between the expression of an inference license in Frege and the corresponding universally quantified conditional.

[21] "Our postulate cannot refer to any such external procedure [as actually drawing lines]. It refers rather to something conceptual. But what is here in question is not a subjective, psychological possibility, but an objective one" (Frege 1914, p. 207).

the activity of writing, independently of drawing lines in a Euclidean diagram and of writing theorems in *Begriffsschrift*. Nevertheless, as Frege indicates, the *discovery* of that truth is not in the same way independent of the activity of writing. Demonstrations extend our knowledge not by creating truths but by showing what can be derived on the basis of given starting points. The possibility of such a showing is an objective possibility; nevertheless, the showing is needed if we are to come to see what is in this way available to be seen.[22]

A related concern arises in light of the status of definitions as stipulations: if a definition is merely a stipulation then it can seem that a conclusion derived from definitions is not properly speaking an objective truth, or at least not a very interesting objective truth but merely something provable given the stipulation. But this too is mistaken. Although a definition is a stipulation, as Frege says, it is also, along another dimension, something about which we can be right or wrong. One cannot be wrong to stipulate that some newly introduced sign is to have the same meaning as some other collection of signs. But one *can* be wrong to think that one has in that collection of signs set out the inferentially articulated content of some concept. Concepts, which are the *Bedeutungen* of concept words, are something objective on Frege's view; it is not up to us to decide what concepts there are, even in mathematics and logic. One's proposed definition can fail to designate any concept. It is, then, a fully objective matter what logical bonds actually obtain among concepts. We can make mistakes, even in mathematics and in logic. In particular, we can think that we have a proof of some theorem when in fact, as we may eventually discover, the conceptions on the basis of which the "proof" proceeds are flawed.

Judgment, conceived following Frege as an acknowledgement of a true thought, can succeed only if the relevant thought *is true*; and inference similarly, which aims to acknowledge a truth on the basis of another truth, can succeed only if the ground of the inference is true *and* the passage is legitimate. But if so, then it can happen that although it might seem to a thinker that a logical bond among concepts has been revealed in the course of a proof such as Frege's, in fact it has not. The possibility, or impossibility, of showing that those bonds obtain is not in any way dependent on the proof; and it is this possibility, or impossibility, that insures that even deductive reasoning from definitions is answerable to something outside of it. One *can* extend one's knowledge by reasoning on the basis of definitions by logic alone precisely because the truth that is revealed in the proof is in this way independent of what we do, including what we do in stipulating in a definition.

22 Is it significant in this context that Frege holds that geometry is synthetic although arithmetic and logic are not? I do not see that it is. To say that geometry is synthetic is, for Frege, to say that its proofs rely on truths that "are not of a general logical nature, but belong to the sphere of some special science", namely, in this case to the sphere of geometry (Frege 1884, §3). Frege's proof of theorem 133 is not synthetic in this sense but instead analytic; it relies "only on general logical laws and definitions" (ibid.). Yet it is, Frege holds, ampliative "and ought therefore, on Kant's view, to be regarded as synthetic" (Frege 1884, §88). What Frege's work reveals, then, is that Kant's distinction has been superseded. The question whether a chain of reasoning can extend our knowledge must be kept separate from the question whether the conclusion is analytic or synthetic in Frege's sense, that is, whether it depends on logic or definitions alone or also on the non-logical laws of some special science.

6 Conclusion

In Frege's concept-script, we have seen, definitions are necessary if one is to prove something about the defined concepts with which one begins. One needs the defined sign, the *definiendum*, if one is to prove something about the particular concepts one cares about; but one needs also the *definiens* if the needed logical bonds are to be forged. Only a proof that begins with definitions, and indeed with fruitful definitions, definitions that are not merely truth-functional, can be ampliative. But it is equally true that only within a proof are definitions of any interest; because definitions are stipulations, the judgments that follow from them taken one by one are, we have seen, utterly trivial. It is only definitions and proofs working together that can yield something new. Only *within* a proof can the peculiar power that resides in definitions by their nature as stipulations regarding simple and complex signs be harnessed to realize something new. In a slogan: proofs without definitions are empty, merely the aimless manipulation of signs according to rules; and definitions without proofs are, if not blind, then dumb. Only a proof can realize the potential of definitions to speak to one another, to pool their resources so as to realize something new.

Frege's *Begriffsschrift* proof of theorem 133 on the basis of his four definitions is, then, ampliative although his derivations in Part II are not. Frege's proof extends our knowledge by showing us how to generate the desired theorem, and thereby reveals a certain logical bond among the defined concepts. And this is made possible, I have argued, by the peculiar nature of Frege's notation, by the fact that a definition formulates, that is, shows (or at least purports to show) the inferentially articulated content of a concept rather than merely saying what it is. Much as a Euclidean diagram formulates the contents of geometrical concepts in a way that enables a course of reasoning that reveals various logical relations among those concepts, so a definition in Frege formulates the contents of mathematical concepts generally in a way that enables a course of reasoning revealing various logical relations among concepts. Although all the details are different, Frege, like Euclid, puts the content of a concept before one's eyes in a two-dimensional array, a kind of diagram, and does so in a way that enables reasoning from the content of that concept *in* the system of signs. As Euclid's system of diagrams is, Frege's is a language *within which* to reason from the contents of concepts in mathematics and to discover thereby truths that extend our knowledge. Just as Frege says (1884, §88), in a *Begriffsschrift* proof from fruitful definitions the theorem is contained in the definitions not as beams are contained in a house, that is, implicitly, needing only to be made explicit, but potentially, as a plant is contained in the seed.

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