Handout for Week 8—Frege in Context

Course materials available at: https://spaceofreasons.netlify.app/courses/frege2025/

Plan: Using logical tools to elaborate *old* concepts-and-claims into *new* ones—
in different senses of 'new':

1. Boolean combinations of microfeatures. Iterate and build to the sky.
2. First bridge: Monadic—>polyadic simple predicates.
3. Second bridge: Functional analysis: simple predicates —> complex predicates.

Quantifiers taking complex predicates to sentences.

4. Cycle of composition and decomposition. Iterate and build to the sky.
5. Third bridge: single premise to multipremise implications.

Alternating quantifiers. Context of the point in Kant and Euclid.

6. Comparing Euclidean roles with Fregean senses as constellations of modes of presentations (functions-applied-to-arguments) that determine patterns of inference.
7. Fourth bridge: Abstraction as introducing new sortal concepts and referring terms.

Frege:

[Kant] seems to think of concepts as defined by giving a simple list of characteristics in no special order; but of all ways of forming concepts, that is one of the least fruitful [früchtbar]. If we look through the definitions given in the course of this book, we shall scarcely find one that is of this description. The same is true of the really fruitful definitions in mathematics, such as that of the continuity of a function. What we find in these is not a simple list of characteristics; every element is intimately, I might almost say organically, connected with the others...
If we represent the concepts (or their extensions) by figures or areas in a plane, then the concept defined by a simple list of characteristics corresponds to the area common to all the areas representing the defining characteristics; it is enclosed by segments of their boundary lines. With a definition like this ...we...use the lines already given in a new way for demarcating an area. [note: Similarly, if the characteristics are joined by "or".] Nothing essentially new, however, emerges in the process. But the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. **What we shall be able to infer from it, cannot be inspected in advance; here we are not simply taking out of the box again what we have just put into it.** The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic.(p. 100-101 [GL])

The truth is [the conclusions drawn from fruitful definitions] are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house. **Often we need several definitions for the proof of some proposition, which consequently is not contained in**

any one of them alone, yet does follow purely logically from all of them together. ([GL] p. 101)

It is the fact that attention is principally given to **this sort of formation of new concepts from old ones**, while other more fruitful ones are neglected which surely responsible for the impression one easily gets in logic that for all our to-ing and fro-ing we never really leave the same spot. ([BLC] p.34)

If we compare what we have here with the definitions contained in our examples, of the continuity of a function and of a limit [and the ancestral of a relation], we see that there's no question there of using the boundary lines of concepts we already have to form the boundaries of the new ones. Rather, totally new boundary lines are drawn by such definitions-and these are the scientifically fruitful ones. Here too**, we use old concepts to construct new ones**, but in so doing we combine the old ones together in a variety of ways by means of the signs for generality, negation, and the condition- al. ([BLC] p.34)

Plucker's insight may be characterized in linguistic terms by considering the equation of a line (or point). Say that a line is determined in a plane by Ax + By + C = 0. **From the form ux + vy + C = 0 alone it is not determined what is to be constant and what is to vary. Taking "u" and "v" (the "line places") to be constant fixes the basic entities**: If we now introduce the coordinates of a straight line into its equation, the equation becomes: (1) ux + vy + 1 = 0 The significance (bedeutung) of this equation can now be expressed better as follows: namely that it indicates the combination of the position of the point x,y with the line u,v; that is the configuration in which the point lies on the line or the line goes through the point, we take either the coordinates of the points or the coordinates of the line as variable. In the first case, what we have in (1) is the equation of a line…
in the other case the equation of a point.. .through suitable choice of coordinates line one has obtained the equation which describes the combined configuration basic elements (grundelementen) with respect to one intuition and the basic elements with respect to the other. These basic elements appear in symmetric relation to one another. The point therefore plays the same role in the geometry of a line as the line plays in the geometry of points.
(Clebsch/Lindemann [1876] p. 29—quoted by Jamie Tappenden "Extending Knowledge and 'Fruitful Concepts'" *Nous* 1995.)

Deduction-Detachment (DD):	Γ, A |~ B, Δ ========= Γ |~ A→B, Δ.	Bidirectional Meta-Inference Line
Incoherence-Incompatibility (II):	Γ, A |~ Δ ========= Γ |~ ¬A, Δ.	Bidirectional Meta-Inference Line

Concepts-and-claims formed by complex predicates can be thought of as formed by a four-stage process.

• First, put together simple predicates and singular terms, to form a set of sentences, say $\{Rab, Sbc, Tacd\}$.

• Then apply sentential **compounding** operators to form more complex sentences, say $\{Rab \rightarrow Sbc, Sbc \& Tacd\}$.

• Then **analyze** by discerning functions by substituting variables for some of the singular terms (individual constants), to form complex predicates, say $\{Rax \rightarrow Sxy, Sxy \& Tayz\}$.

• Finally, apply quantifiers to bind some of these variables, to form new sentences and complex predicates, for instance the one-place predicates (in y and z) $\{\exists x[Rax \rightarrow Sxy], \forall x \exists y[Sxy \& Tayz]\}$.

If one likes, this process can now be repeated, with the complex predicates just formed playing the role that simple predicates originally played at the first stage, yielding the new sentences $\{\exists x[Rax \rightarrow Sxd], \forall x \exists y[Sxy \& Taya]\}$.

They can then be conjoined, and the individual constant a substituted for to yield the further one-place complex predicate (in z): $\exists x[Rzx \rightarrow Sxd] \& \forall x \exists y[Sxy \& Tzyz]$.

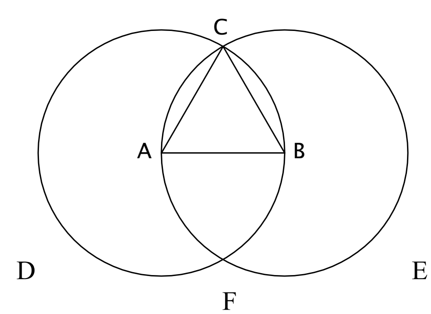
We can use these procedures to build to the sky, repeating these stages of concept construction by sequential **composition** and **decomposition** as often as we like.

Frege's rules tell us how to compute the inferential roles of the concepts formed at each stage, on the basis of the inferential roles of the raw materials, and the operations applied at that stage. This is the heaven of concept formation he opened up for us.

Kant's challenge: to understand *multipremise* inferences.

How can a *set* of premises jointly *contain* (in the sense of *imply*) something that is not *contained in* any of the premises individually? Where does the extra content come from?

Diagram of proposition I.1 of Euclid's *Elements*.



For Frege, the modes of presentation contained in the sense of one expression both consist of functions-applied-to-arguments into which an expression such as '2^4' can be analyzed and each correspond to a pattern of inferences (implications and incompatibilities) relating the expressions to their substitutional variants.

Bolzano's strategy is to understand the *modes of combination* [*Verbindungsarten*] of expressions—the surplus, above the mereological 'content' that consists in the union of their parts, and which distinguishes 'learned son of an ignorant father' from 'ignorant son of a learned father'—in terms of *relations* of expressions to other expressions. Those relations that functionally define modes of combination are semantic invariances under substitution (his 'variation'), each codifying a pattern of inferences.