Handout for Week 5

Passages from “Function and Concept”

<u>Functions are ‘incomplete’ or ‘unsaturated’:</u>

I am concerned to show that the argument does not belong with the function, but goes together with the function to make up a complete whole; for the function by itself must be called incomplete, in need of supplementation, or 'unsaturated.' And in this respect functions differ fundamentally from numbers. Since such is the essence of the function, we can explain why, on the one hand, we recognize the same function in $2*1^3 +1$ and $2*2^3 +2$ even though these expressions stand for different numbers, whereas, on the other hand, we do not find one and the same function in $2*1^3 +1$ and 4-1, in spite of their equal numerical values. Moreover, we now see how people are easily led to regard the form of the expression as what is essential to the function. We recognize the function in the expression by imagining the latter as split up, and the possibility of thus splitting it up is suggested by its structure. [24]

<u>Values, arguments, and functions</u>:

We give the name 'the value of a function for an argument' to the result of completing the function with the argument. [25]

<u>Ranges or courses of values</u>:

If we compare with this the function $x*(x - 4)$ [with x^2+4x] we find that they have always the same value for the same argument….I express this as follows: the function $x*(x - 4)$ has the same **range of values** as the function x^2+4x. [26]

<u>Ranges of values as formed by ***abstraction***</u>:

If we write ‘$x^2 – 4x = x*(x - 4)$, ' we have not put one function equal to the other, but only the values of one equal to those of the other. And if we so understand this equation that it is to hold whatever argument may be substituted for x, then we have thus expressed that an equality holds generally. But we can also say: 'the value-range of the function $x*(x - 4)$ is equal to that of the function x^2-4x and here we have an equality between ranges of values. The possibility of regarding the equality holding generally between values of functions as a [particular] equality, viz. an equality between ranges of values, is, I think, indemonstrable; it must be taken to be a fundamental law of logic. [26]

Accordingly, e.g., $’\varepsilon(\varepsilon^2-4\varepsilon)$ is the value-range of the function x^2-4x. [27]

<u>Introducing *truth values* as objects, by abstraction</u>:

I begin by adding to the signs +, -etc., which serve for constructing a functionalp. 13] expression, also signs such as **=**, **>**,**<**, so that I can speak, e.g., of the function $x^2=1$, where *x* takes the place of the argument as before. The first question that arises here is what the values of this function are for different arguments. Now if we replace x successively by -1, 0, 1, 2, we get:

$(-1)^2 =1$,

$(0)^2 = 1$,
$(1)^2 = 1$,
$(2)^2 = 1$.

Of these equations the first and third are true, the others false. I now say: 'the value of our function is a truth-value' and distinguish between the truth-values of what is true and what is false. I call the first, for short, the True; and the second, the False. Consequently, e.g., ‘$2^2=4$’ stands for the True as, say, ‘2^2’ stands for 4. And ‘$2^2=1$’ stands for the False. Accordingly ‘$2^2=4$’, ‘$2>1$’, ‘$2^4=4^2$’, stand for the same thing, viz. the True, so that in $(2=4) = (2>1)$ we have a correct equation. [29]

Sense and reference, with paradigm being different functions computing the same value.

We see from this that from identity of reference there does not follow identity of the thought [expressed]. If we say 'the Evening Star is a planet with a shorter period of revolution than the Earth,' the thought we express is other than in the sentence 'the Morning Star is a planet with a shorter period of revolution than the Earth'; for somebody who does know that the Morning Star is the Evening Star might regard one as true and the other as false. And yet both sentences must have the same reference; for it is just a matter of interchanging the words 'Evening Star' and 'Morning Star,' which have the same reference, i.e. are proper names of the same heavenly body. We must distinguish between sense and reference. ‘2^4’ and ‘4^2’ certainly have the same reference, i.e. they are proper names of the same number; but they have not the same sense; consequently, ‘$2^4 = 4^2$’ and ‘$4*4=4^2$’ have the same reference, but not the same sense (which means, in this case: they do not contain the same thought). [29]

Concepts are functions whose values are truth values.

We saw that the value of our function $x^2=1$ is always one of the two truth-values. Now if for a definite argument, e.g. -1, the value of the function is the True, we can express this as follows: 'the number -1 has the property that its square is 1'; or, more briefly, ' -1 is a square root of 1'; or, ' -1 falls under the concept: square root of 1.’ If the value of the function $x^2=1$ for an argument, e.g. for 2, is the False, we can express this as follows: ‘2 is not a square root of 1’ or ‘2 does not fall under the concept: square root of 1.' We thus see how closely that which is called a concept in logic is connected with what we call a function. Indeed, we may say at once: **a concept is a function whose value is always a truth-value**. [30]

Definition of ‘extension of a concept’, vindicating the objectionable footnote in *GL*.

Hence we can designate as an extension the value-range of a function whose value for every argument is a truth-value. [31]

From equations to statements:

The linguistic form of equations is a statement.

A statement contains (or at least purports to contain) a thought as its sense; and this thought is in general true or false; i.e. it has in general a truth-value, which must be regarded as the reference of the sentence, just as (say) the number 4 is the reference of the expression ‘2+2’ or London of the expression 'the capital of England.' [31]

Discerning functions in statements:

Statements in general, just like equations or inequalities or expressions in Analysis, can be imagined to be split up into two parts; one complete in itself, and the other in need of supplementation, or 'unsaturated.' [31]

Next generalization of function: allow objects in general as arguments and values.

We see that here we have undertaken to extend [the application of the term] in the other direction, viz. as regards what can occur as an argument. Not merely numbers, but objects in general, are now admissible; and here persons must assuredly be counted as objects. The two truth-values have already been introduced as possible values of a function; we must go further and admit objects without restriction as values of functions. [31]

Objects vs. Functions, Truth-values as objects:

When we have thus admitted objects without restriction as arguments and values of functions, the question arises what it is that we are here calling an object…An object is anything that is not a function, so that an expression for it does not contain any empty place.
A statement contains no empty place, and therefore we must regard what it stands for as an object. But what a statement stands for is a truth-value. Thus the two truth-values are objects. [32]

Value-ranges and extensions are objects:

Value-ranges of functions are objects, whereas functions themselves are not. We gave the name 'value-range' also to ’$\varepsilon(\varepsilon^2=1)$, but we could also have termed it the extension of the concept: square root of I. Extensions of concepts likewise are objects, although concepts themselves are not. [32]

Concepts must have sharp boundaries, functions must have values for every argument:

…the requirement as regards concepts, that, for any argument, they shall have a truth-value as their value; that it shall be determinate, for any object, whether it falls under the concept or not. In other words: as regards concepts we have a requirement of sharp delimitation…
The requirement of the sharp delimitation of concepts thus carries along with it this requirement for functions in general that they must have a value for every argument. [33]

Truth functions: both arguments and values of function are truth-values:

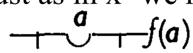
We have so far considered truth-values only as values of functions, not as arguments…
But now we must deal with certain functions that are of importance to us precisely when their argument is a truth-value. [33]

Generality:

I now take the sign $-\!\overset{\mathfrak{a}}{\smile}\!-f(\mathfrak{a})$ to mean the True when the function f(x) always has the True as its value, whatever the argument may be; in all other cases $-\!\overset{\mathfrak{a}}{\smile}\!-f(\mathfrak{a})$ is to stand for the False. [35]

Functions whose arguments are functions:

Now just as in x^2 we have a function whose argument is indicated by ‘x’, I also conceive of



as the expression of a function whose argument is indicated by *'f'*.

Such a function is obviously a fundamentally different one from those we have dealt with so far; for only a function can occur as its argument.

Now just as functions are fundamentally different from objects, so also functions whose arguments are and must be functions are fundamentally different from functions whose arguments are objects and cannot be anything else. I call the latter first-level, the former second-level, functions.

In the same way, I distinguish between first-level and second-level concepts. [38]

…The ontological proof of God's existence suffers from the fallacy of treating existence as a first-level concept. [38, footnote]

Grasping functions requires generalizing:

If we look back from here over the development of arithmetic, we discern an advance from level to level. At first people did calculations with individual numbers, 1, 3 etc. 2+3=5, 2*3=6 are theorems of this sort. Then they went on to more general laws that hold good for all numbers. What corresponds to this in symbolism is the transition to the literal notation….A theorem of this sort is (a+b)*c = a*c + b*c.

At this stage they had got to the point of dealing with individual functions; but were not yet using the word, in its mathematical sense, and had not yet formed the conception of what it now stands for.

The next higher level was the recognition of general laws about functions, accompanied by the coinage of the technical term 'function.' What corresponds to this in symbolism is the introduction of letters like *f*, *F*, to indicate functions indefinitely….

Now at this point people had particular second-level functions, but lacked the conception of what we have called second-level functions...By forming that, we make the next step forward. [41]

The first place where a scientific expression appears with a clear-cut reference is where it is required for the statement of a law. This case arose, as regards the function, upon the discovery of higher Analysis. Here for the first time it was a matter of setting forth laws holding for functions in general. [21]