Handout for Week 4

https://spaceofreasons.netlify.app/courses/frege2025/

Plan:

1. Recap on Counting, Sortals, and Identity
2. Botanizing Identities by Sortals
3. Singular Terms from Sortals: Definite Descriptions
4. Abstraction Basics: Treating Equivalence Relations as Identities
5. On the Concept Abstract Object: Beware of the Platonic Principle
6. Abstraction and Cross-Sortal Identity: the Julius Caesar Problem
7. Numbers Basics: One-to-One Correspondence as the Equivalence Relation
8. Sketch of some Mathematical Elaborations of Abstraction:
   Set Theory, Mereology, Order Isomorphisms, Galois Connections, and Beyond
9. Frege's Substitutional Analysis by Functions forms Sortals by Abstraction

Three kinds of identities, distinguished by the relations between their sortals

1) **Intrasortal identities**: 'The inventor of bifocals is (=) the first Postmaster General.'
Identities involving both singular terms are governed by the *same* sortal: person (or human being).
If the inventor of bifocals is (=) the first Postmaster General, then they are the same *person*.

2) **Weakly cross-sortal identities**: 'Pedro the donkey is (=) the only mammal in the stable.'
In weakly cross-sortal identities, the relations between the sortals governing the singular terms is ***Linnaean***: We can reason *down* from the identity and individuation of the Linnaean *genus* to the identity and individuation of its *species*.
All Ks are Js, and if *a* and *b* are both Ks, and they are the *same* Js then they are the *same* Ks.
If *a* and *b* are donkeys, and *a* is the *same mammal* as *b*, then *a* is the same *donkey* as *b*.

3) **Strongly cross-sortal identities: '**Goliath, the statue is (=) Lumpl, the lump of clay.'
(Clay) statue and lump of clay are *different* sortals: not all lumps of clay are statues.
They are *not* related in the *Linnaean* way:
***If *a* and *b* are statues, and *a* is the same lump of clay as *b*, then *a* is the same *statue* as *b*.**
This does *not* hold. At time *t* Lumpl might have been shaped into the statue Goliath, and at time *t'* Lumpl might have been shaped into the statue David.

Claim: **Strongly cross-sortal identities**, that is, those between singular terms governed by different non-Linnaeanly related sortals (excluding purely mathematical ones) **are never true**.

Test cases: i) kitten/cat vs. passenger/person.
ii) fridgeons?

Definite descriptions form *singular terms* from *sortals*.
To be entitled to introduce a singular term of the form '*the* K *x* such that F(x)' one must prove both *existence* and *uniqueness*: that there *is* a K that is F, and that there is *only one* K that is F.

> **The definition of an object in terms of a concept under which it falls** is a very different matter. For example, the expression "the largest proper fraction" has no content, since **the definite article claims to refer to a definite object**. On the other hand, the concept "fraction smaller than 1 and such that no fraction smaller than one exceeds it in magnitude" is quite unexceptionable: in order, indeed, to prove that there **exists** no such fraction, we must make use of just this concept, despite its containing a contradiction. If, however, **we** wished to use this concept for defining an object falling under it, it would, of course be necessary first to show two distinct things:
>
> - that some object falls under this concept;
> - that only one object falls under it.
>
> Now since the first of these propositions, not to mention the second, is false, it follows that the expression "the largest proper fraction" is senseless. [§74, footnote]

Equivalence relations are reflexive, symmetric, and transitive.
Reflexive: $\forall x[xRx]$
Symmetric: $\forall x,y[xRy \Leftrightarrow yRx]$
Transitive: $\forall x,y,z[(xRy \,\&\, yRz) \Rightarrow xRz]$

Identity claims, whose conceptual content (*begrifflich Inhalt*) is understood in terms of their licensing intersubstitution *salva consequentia*, express an equivalence relation.

Abstraction is defining new sortals from old by taking or treating an *equivalence relation* on singular terms governed by the old (relatively concrete) sortal as an *identity* between singular terms governed by the new (relatively abstract) sortal.

> § 64. The judgement "line *a is* parallel to line b", or, using symbols, $a \parallel b$, can be taken as an identity. If we do this, we obtain the concept of direction, and say: "the direction of line *a* is identical with the direction of line *b*".

Some further passages from the *Grundlagen*:

If we were to try saying "q is a direction if it is introduced [eingeführt] by means of the definition set out above, then we should be treating the way in which the object q is introduced as a property of q, which it is not. The definition of an object does not, as such, really assert anything about the object, but only lays down the meaning of a symbol. After this has been done, the definition transforms itself into a judgement, which does assert about the object; but now it no longer introduces the object, it is exactly on a level with other assertions made about it. If, moreover, we were to adopt this way out, we should have to be presupposing that an object can only be given in one single way [dass ein Gegenstand nur auf eine einzige Weise gegeben werden könnte]; for otherwise it would not follow, from the fact that q was not introduced by means of our definition, that it could not have been introduced by means of it. All identities would then amount simply to this, that whatever is given to us in the same way is to be reckoned as the same. This, however, is a principle so obvious and sterile [so selbstverständlich und so unfruchtbar] as not to be worth stating. We could not, in fact, draw from it any conclusion which was not the same as one of our premises. Why is it after all that we are able to make use of identities with such significant results in such diverse fields? Surely it is rather because we are able to recognize something as the same again even though it is given in a different way [dass man etwas wiedererkennen kann, obwohl es auf verscheidene Weise gegeben ist]. [67]
In the same way with the definitions of fractions, complex numbers and the rest, everything will in the end come down to **the search for a judgement-content [beurtheilbaren Inhalt] which can be transformed [verwandelt] into an identity** whose sides precisely are the new numbers. In other words, what we must do is fix the sense of a recognition-judgement for the case of these numbers. [104]

For every object there is one type of proposition which must have a sense, namely the recognition-statement, which in the case of numbers is called an identity...The problem, therefore, was this: to fix the sense of a numerical identity... [106]

When are we entitled to regard a content as that of a recognition-judgement? For this a certain condition has to be satisfied, namely that it must be possible in every judgement to substitute without loss of truth the right-hand side of our putative identity for its left-hand side. [107]

Here, just as there, it is a matter of fixing the content of a recognition-judgement [Wiedererkennungsurtheils] [109]

We can, of course, by using the last two definitions together, say what is meant by "the number I + x belongs to the concept F" and then, using this, give the sense of the expression "the number 1 + 1 + x belongs to the concept F" and so on; but we can never·-to take a crude example decide by means of our definitions whether any concept has the number JULIUS CAESAR belonging to it, or whether that same familiar conqueror of Gaul is a number or is not. [56]

It will not, for instance, decide for us whether England is the same as the direction of the Earth's axis-if I may be forgiven an example which looks nonsensical. Naturally no one is going to confuse England with the direction of the Earth's axis; but that is no thanks to our definition of direction .. That says nothing as to whether the proposition "the direction of *a* is identical with *q″* should be affirmed or denied, except for the one case where *q* is given in the form of "the direction of *b″*. What we lack is the concept of direction; for if we had that, then we could lay it down that, if *q* is not a direction, our proposition is to be denied, while if it is **a** direction, our original definition will decide whether it is to be denied or affirmed. So the temptation is to give as our definition:
q is a direction, if there is a line *b* whose direction is *q*.
But then we have obviously come round in a circle. For in order to make use of this definition, we should have to know already in every case whether the proposition " *q* is identical with the direction of *b″* was to be affirmed or denied. [66]

My definition is therefore as follows:
the Number which belongs to the concept F is the extension[1] of the concept "equal to the concept F". [69]

[1] I believe that for "extension of the concept" we could write simply "concept". But this would be open to the two objections:
1. that this contradicts my earlier statement that the individual numbers are objects, as is indicated by the use of the definite article in expressions like "the number two" and by the impossibility of speaking of ones, twos, etc. in the plural, as also by the fact that the number constitutes only an element in the predicate of a statement of number;
2. that concepts can have identical extensions without thcn1selves coinciding.
I am, as it happens, convinced that both these objections can bé met; but to do this would take us too far afield for present purposes. **I assume that it is known what the extension of a concept is**. [69]