Handout for Week 2: *Begriffsschrift*

Frege in Context—Bob Brandom—Fall 2025
https://spaceofreasons.netlify.app/courses/frege2025/

§3 …the contents of two judgements can differ in two ways:
either the conclusions that can be drawn from one when combined with certain others also always follow from the second when combined with the same judgements or else this is not the case.
The two propositions 'At Plataea the Greeks defeated the Persians' and 'At Plataea the Persians were defeated by the Greeks' differ in the first way. Even if a slight difference in sense can be discerned, the agreement predominates.
Now **I call that part of the content that is the *same* in both the *conceptual content*** [begriffliche Inhalt]...
[O]*nly this* has significance for the *Begriffsschrift*…

[I]*n my formula language…the only thing that is relevant in a judgement is that which influences its possible* ***consequences****.*
Everything that is necessary for a valid* [richtig Schluss] *inference is fully expressed*; but what is not necessary is mostly not even indicated; nothing is left to guessing.* [§3]

In contrast we may now set out the aim of my concept-script.
Right from the start I had in mind the *expression of a content.*
What I am striving after is a *lingua characterica* in the first instance for mathematics, not a *calculus* restricted to pure logic.
But **the content is to be rendered more exactly** than is done by verbal language. [12]

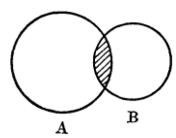
The reason for this inability to form concepts in a scientific manner lies in the lack of one of the two components of which every highly developed language must consist. That is, we may distinguish the formal part which in verbal language comprises endings, prefixes, suffixes and auxiliary words, from **the material part** proper. The signs of arithmetic correspond to the latter. What we still lack is the logical cement that will bind these building stones firmly together….
In contrast, Boole's symbolic logic only represents the formal part of language, and even that incompletely. [13]

It seems to me to be even easier to extend the domain of this formula language to **geometry**. Only a few more symbols would have to be added for the intuitive relations that occur here. In this way one would obtain a kind of *analysis situs.*

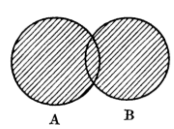
The transition to **the pure theory of motion** and thence to **mechanics** and **physics** might follow here. In the latter fields, **where besides conceptual necessity, natural necessity prevails**, a further development of the symbolism with the advancement of knowledge is easiest to foresee. But that is no reason to wait until such advancement appears to have come to an end.
[Preface, Beaney p. 50.]

As opposed to this, I start out from judgements and their contents, and not from concepts. **The precisely defined hypothetical relation between contents of possible judgement has a similar significance for the foundation of my concept-script to that which identity of extensions has for Boolean logic.**
I only allow the formation of concepts to proceed from judgements. [16]

Now it is worth noting in all this, that in practically none of these examples is there first cited the genus or class to which the things falling under the concept belong and then the characteristic mark of the concept, as when you define 'homo' as *"animal rationale'.* Leibniz has already noted that here we may also conversely construe *'rationale'* as genus and *'animal'* as species. In fact, by this definition *'homo'* is to be whatever is *'animal'* as well as being *·rationale'.*



If the circle *A* represents the extension of the concept *'animal'* and *B* that of *'rationale',* then the region common to the two circles corresponds to the extension of the concept *·homo'.* And it is all one whether I think of that as having been formed from the circle *A* by its intersection with *B* or vice versa. This construction corresponds to logical multiplication. Boole would express this, say, in the form $C = AB$, where *C* means the extension of the concept *'homo'.* You may also form concepts by logical addition.



We have an example of this if we define the concept 'capital offence' as murder or the attempted murder of the Kaiser or of the ruler of one's own *Land* or of a German prince in his own *Land.* The area *A* signifies the extension of the concept 'murder', the area *B* that of the concept 'attempted murder of the Kaiser or of the ruler of one's own *Land* or of a German prince in his own *Land'.* Then the whole area of the two circles, whether they have a region in common or not, will represent the extension of the concept 'capital offence'.

If we look at what we have in the diagrams, we notice that in both cases the boundary of the concept, whether it is one formed by logical multiplication or addition is made up of parts of the boundaries of the concepts already given. This holds for any concept formation that can be represented by the Boolean notation. This feature of the diagrams is naturally an expression of something inherent in the situation itself, but which is hard to express without recourse to imagery. In this sort of concept formation, one must, then, assume as given a system of concepts, or speaking metaphorically, a network of lines. These really already contain the new concepts: all one has to do is to use the lines that are already there to demarcate complete surface areas in a new way. It is the fact that attention is principally given to this sort of formation of new concepts from old ones, while other more fruitful ones are neglected which surely is responsible for the impression one easily gets in logic that for all our to-ing and fro-ing we never really leave the same spot. [33-4]

If we compare what we have here with the definitions contained in our examples, of the continuity of a function and of a limit, and again that of following a series which I gave in §26 of my *Begriffsschrift,* we see that there's no question there of using the boundary lines of concepts we already have to form the boundaries of the new ones. Rather, **totally new boundary lines are drawn by such definitions-and these are the scientifically fruitful ones.** Here too, we use old concepts to construct new ones, but in so doing we combine the old ones together in a variety of ways by means of the signs for generality, negation and the conditional. [34]

I believe in this essay I have shown:
(1) My concept-script has a more far-reaching aim than Boolean logic, in **that it strives to make it possible to present a content** when combined with arithmetical and geometrical signs.
(2) Disregarding content, within the domain of pure logic it also, thanks to the notation for generality, commands a somewhat wider domain than Boole's formula-language.
…
(4) It is in a position to represent the formations of the concepts actually needed in science, in contrast to the relatively sterile multiplicative and additive combinations we find in Boole. [46]

If, in an expression (whose content need not be a judgeable content), a simple or complex symbol occurs in one or more places, and we think of it as replaceable at all or some of its occurrences by another symbol (but everywhere by the same symbol), then we call the part of the expression that on this occasion appears invariant the ***function****, and the replaceable part its* ***argument****.* [§9]

For us the different ways in which the same conceptual content can be taken as a function of this or that argument has no importance so long as function and argument are fully determined. But **if the argument becomes *indeterminate*** as in the judgement 'You can take as argument for "being representable as the sum of four squares" whatever positive whole number you like: the proposition always remains correct', **then the distinction between function and argument acquires significance with regard to *content.*** [§9]

One sees here particularly clearly that **the concept of function** in Analysis, which in general I have followed, is far more restricted than that developed here. . [§10]

But of course subtraction is a proper function. What is distinctive of it, and of the other inverse operations, is that to see that it is requires seeing the sentence '+(2,3) = 5' not as having inherently the form of an identity but instead as presenting, by means of familiar arithmetical symbols, an arithmetical relation among three numbers, one that can be carved up in various ways into function and argument. **To understand subtraction, that is, one learns to see the sentence '+(2,3) = 5' in a radically new way. One learns to *read* it differently, not as the result of a stepwise process (first take two and three and apply the plus function to them, then take the result and set it equal to five), but simply as exhibiting an arithmetical relationship between two, three, and five, one that can be analyzed in a variety of ways.** [Macbeth, *Frege's Logic*, p. 42]

The critical feature of a sentence such as '$2^4 = 16$' in the formula language of arithmetic as it is now being conceived is that it merely presents three numbers in an arithmetical relation and is variously analyzable. [Macbeth, *Frege's Logic*, p. 42]

A sentence of the formula language of arithmetic such as '$2^4 = 16$' can be carved up in various ways into function and argument to yield a sentence that ascribes a concept to a number.
On one analysis, it says that two is a fourth root of sixteen, on another that four is a logarithm of sixteen to the base two, and so on. [Macbeth, *Frege's Logic*, p. 40]

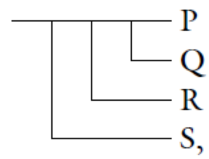
Similarly, we have suggested, **we can learn to read a sentence of the formula language of arithmetic such as '$2^4 = 16$' not from left to right (or right to left), but simply as a presentation of the numbers two, four, and sixteen in a certain arithmetical relation, one that can be read, analyzed into function and argument, in a variety of ways**.
[Macbeth, *Frege's Logic*, p. 43]

Only relative to an analysis that identifies some number(s) as argument(s) and the remainder as the function is the sentence so read correctly described as saying something *about* something. [Macbeth, *Frege's Logic*, p. 43]

In the 1879 logic Frege claims that a Begriffsschrift sentence presents "a mere combination of ideas {blosse Vorstellungsverbindung}" §2 [Macbeth, *Frege's Logic*, p. 45]

Frege's conditional stroke is his sign for the primitive logical relation, and it is that stroke that gives his logical language its peculiar two-dimensional character.
[Macbeth, *Frege's Logic*, p. 46]

Because this content that is common to such equipollent propositions "alone is of concern to logic," "all that would be needed [in an adequate logic] would be a single standard proposition for each system of equipollent propositions" (*PMC* 67). Frege's two-dimensional notation provides just such a standard proposition for the case of conditionals with more than one condition. [Macbeth, *Frege's Logic*, p. 50]



Each of '$S \supset (R \supset (Q \supset P))$', '$S \supset ((R \& Q) \supset P)$', '$(S \& R) \supset (Q \supset P)$', and '$(S \& R \& Q) \supset P$' (or their natural language equivalents) represents in this way one path through Frege's two-dimensional structure, one perspective it is possible to take on it. The equivalence of these four formulae, though it must be proven in standard (one-dimensional) notation, is a given of Frege's two-dimensional notation. [Macbeth, *Frege's Logic*, p. 51]

In a linear notation, on the standard reading of it, there is always a main connective. [Macbeth, *Frege's Logic*, p. 51]

In our standard notations, by contrast, having proved that, say, '$(P \& Q) \supset R$' is equivalent to '$(Q \& P) \supset R$' will not save one the trouble of having also to prove that, say, '$P \supset ((Q \& R) \supset S)$' is equivalent to '$Q \supset ((R \& P) \supset S)$'. [Macbeth, *Frege's Logic*, p. 52]