Handout for Week 11

Abstraction and the Julius Caesar Problem in Axiom V of *Grundgesetze*

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Frege's Argument from §10 of *Grundgesetze*:

The course of value of a function F is written as $'\delta(F\delta)$.

a) Axiom V of the *Grundgesetze*: $\mathbf{'\delta(F\delta) = '\alpha(G\alpha)}$ **iff** $\mathbf{(\forall x)[Fx \Leftrightarrow Gx]}$.

Frege recognizes that this principle alone does not suffice to determine the identity of objects which are courses of values. To show this, he points out that if X is a function which yields distinct values if and only if it is applied to distinct arguments (what we may call an "individuation-preserving" function), then:

a') $X('\delta(F\delta)) = X('\alpha(G\alpha))$ iff $(\forall x)\,[Fx\Leftrightarrow Gx]$

without its having been settled, for instance, whether

a") $X('\delta(F\delta)) = '\alpha(G\alpha)$

for any F and G (including the case in which F = G).

Compare: the p-direction of a line as the set of lines *perpendicular* to it, rather than *parallel* to it.

(a) determines only the truth values of *homogeneous* identities, those both terms of which are of the form $'\delta(F\delta)$. And (a') determines only the truth values of identities which are homogeneous in that both terms have the form $X('\delta(F\delta))$. But (a") asks about *heterogeneous* identities, whose terms are of different forms. Another identity which is heterogeneous and whose truth value is accordingly not settled by principle (a) is Julius Caesar = $'\delta(F\delta)$.

To fix up this indeterminateness, which would result from taking Axiom V alone as the definition of courses of values, Frege proposes to supplement it by stipulating the truth values of the heterogeneous identities. The point is that distinct objects which are not given as courses of values are stipulated to be identical to the courses of values of a like number of arbitrary distinct functions. The task is to show that such a stipulation is legitimate.

What is to be shown is that it is legitimate to stipulate (a) above, determining the homogeneous identities involving courses of values, together with the following stipulation for heterogeneous identities:

(b) $'\tau(L\tau) = t_1$ and $'\sigma(M\sigma) = t_2$

where $t_1 \neq t_2$ and $(\exists x)\,(Lx \neq Mx)$.

L and M are to be arbitrary functions, and t_1 and t_2 are terms which are not of the form $'\alpha\,(F\alpha)$.

For the purposes of the *GG* argument, the terms in question are "the True" and "the False."

To start, suppose it has been stipulated that:

(c) $\sim\eta(F\eta) = \sim\gamma(G\gamma)$ iff $(\forall x)\ [Fx \Leftrightarrow Gx]$,

that is, we stipulate the homogeneous identities for terms of the form $\sim\eta(F\eta)$, where the function which associates objects so denominated with functions F is unknown except that principle (c) holds.

The function X is defined by five clauses:

(1) X(Julius Caesar) = $\sim\eta(L\eta)$

(2) X($\sim\eta(L\eta)$) = Julius Caesar

(3) X(England) = $\sim\gamma(M\gamma)$

(4) X($\sim\gamma(M\gamma)$) = England

(5) **For all other y, X(y) = y**.

The function X is constant except when it is applied to either the two objects which are not specified as the result of applying ~-abstraction to some function (Julius Caesar and England, or the True and the False) or to the result of applying ~-abstraction to the arbitrarily chosen functions L and M. In these special cases, the function X simply permutes the distinguished values.

X is constructed to be individuation preserving, so that a correlation is preserved between distinctness of its arguments and distinctness of its values. It follows then that:

(d) $X(\sim\eta(F\eta)) = X(\sim\gamma(G\gamma))$ iff $(\forall x)\ [Fx \Leftrightarrow Gx]$.

In these terms we could now *define* the course of values notation (which has not previously appeared in this argument) by agreeing to let:

(e) **$'\alpha(F\alpha) =_{df} X(\sim\eta(F\eta))$** for all functions F.

Given the definition (e) and the truth of (d), principle (a) for courses of values follows immediately.

The truth of (d) follows from (c), together with clauses (1)-(5) defining the function X.

But clauses (2) and (4) of that definition, together with (e), entail principle (b) concerning courses of values (with the substitution of Julius Caesar for t_l and England for t_2).

Thus, given only the homogeneous identities in (c), we have constructed courses of values in such a way that their homogeneous identities in (a) can be shown to hold *and* in such a way that heterogeneous identities can be *proven* for two of them, since

$'\alpha(L\alpha)$ = Julius Caesar (= $X(\sim\eta(F\eta))$) and

$'\delta(M\delta)$ = England (= $X(\sim\gamma(M\gamma))$).

The legitimacy of stipulating heterogeneous identities in the context of a principle determining homogeneous ones has been shown by reducing the questionable stipulation to the composition of two obviously acceptable forms of stipulation: the specification of the values which the function X is to take for various arguments-in particular in clauses (2) and (4), and the introduction of the expression ' ’α(Fα)' (previously without a use) as a notational abbreviation of ' X(~η(Fη)).'

<u>Objection</u>:
This imaginative argument is Frege's ultimate response defending the use of abstraction in his account of number and of logical objects generally against the “Julius Caesar problem” about heterogeneous identities he had raised but not answered in the *Grundlagen*, and for the intelligibility of his stipulation that *Bedeutung*(⌈t⌉)=t, the name-bearer model.
Seen in that context, the argument is fallacious.

The problem concerns the extremal clause (5) of the definition of the individuation-preserving function X. If that clause is expanded to make explicit what is contained in the condition "for all other y," it becomes:
(5') **(∀y) [(y≠Julius Caesar & y≠~η(Lη) & y≠England & y≠~γ(Mγ) ⇒ X(y)=y].**
It may then be asked whether it is appropriate at this point in the argument to make use of a condition such as **y≠~γ(Mγ)**. If the term substituted for y is also represented as the product of applying ~-abstraction to some function, then clause (c) will settle the truth value of the resulting identity. For it settles just such homogeneous identities.
But what of the case in which the identity is *heterogeneous*?
[Note that in the context of *GG*, there are not (yet) any other objects besides the truth values, so this issue does not arise.]
All that has been fixed concerning ~-abstraction is principle (c), which says nothing about such identities. Indeed, the whole strategy of the argument depends on starting from a specification of purely homogeneous identities with one sort of abstractor (~) and using the function X to construct an abstractor (’) for which the heterogeneous identities are specified.
Nothing which has been said, or, given the strategy just indicated, *could* be said, settles a truth value for heterogeneous identities such as
(f) Julius Caesar = ~γ(Mγ).
and
(g) England = ~η(Lη).
For all that principle (c) concerning ~-abstraction and the distinctness of the functions L and M settle, (f) could be true and (g) false. Given the truth of (f), substituting in clause (4) would yield that X(Julius Caesar)= England, and so by clause (1) that England = ~η(Lη), that is, that (g) is true. So the definition of X presupposes valuations for heterogeneous identities which it is in no way entitled to.
Again, suppose that in (5’) the value of *y* is set as ‘the direction of the Earth’s axis’.
(5’) does not settle whether X(the direction of the Earth’s axis) = the direction of the Earth’s axis.
For we don’t know how to settle whether the direction of the Earth’s axis = ~η(Lη) or ~γ(Mγ).

And the question of whether it is identical to Julius Caesar or England just *is* the original "Julius Caesar problem." For the general case, Frege's argument begs the question.