Passages from:
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Introduction

When we ask someone what the number one is, or what the symbol one means [bedeute], we get as a rule the answer "Why, a thing". [Introduction, p. i]

Yet if everyone had the right to understand by this name whatever he pleased, then the same proposition about one would mean different things for different people,--such propositions would have no common content. [Introduction, p. i]

[I]t is not true that there are different kinds of laws of thought [Denkgesetzen] to suit the different kinds of objects thought about. Such differences as there are consist only in this, that the thought is more or less pure, less dependent or more upon psychological influences and on external aids such as words or numerals, and further to some extent too in the finer or coarser structure of the concepts involved; but it is precisely in this respect that mathematics aspires to surpass all other sciences, even philosophy. [Introduction, iii]

...the concept of number, as we shall be forced to recognize, has a finer structure [feineren Bau] than most of the concepts of the other sciences, even although it is still one of the simplest in arithmetic. [Introduction, iv]

Never let us take a description of the origin of an idea for a definition, or an account of the mental and physical conditions on which we become conscious of a proposition for a proof of it. A proposition may be thought, and again it may be true; let us never confuse these two things. We must remind ourselves, it seems, that a proposition no more ceases to be true when I cease to think of it than the sun ceases to exist when I shut my eyes. [Introduction, vi]

We suppose, it would seem, that concepts sprout in the individual mind like leaves on a tree, and we think to discover their nature by studying their birth... What is known as the history of concepts is really a history either of our knowledge of concepts, or of the meanings of words. Often it is only after immense intellectual effort, which may have continued over centuries, that humanity at last succeeds in achieving knowledge of a concept in its pure form [in seiner Reinheit], in stripping off the irrelevant accretions [den fremden Umhüllungen herauszuschälen] which veil it from the eyes of the mind. [Introduction, p. vii]

In the enquiry that follows, I have kept to three fundamental principles:
always to separate sharply the psychological from the logical, the subjective from the objective;
never to ask for the meaning of a word in isolation, but only in the context of a proposition [im Satzzusammenhang];

never to lost sight of the distinction between concept and object.
In compliance with the first principle, I have used the word 'idea' [Vorstellung] always in the psychological sense, and have distinguished ideas from concepts and from objects. If the second principle is not observed, one is almost forced to take as the meanings of words mental pictures or acts of the individual mind, and so to offend against the second principle as well. As to the third point, it is a mere illusion to suppose that a concept can be made an object without altering it....With numbers of all these types, as with the positive whole numbers, it is a matter of fixing the sense of an identity [den Sinn einer Gleichung festzustellen. [Introduction, p. x]

[BB--The separation in the first case is between factual and normative. He follows Kant in calling the latter (what is binding on us as a law) 'objective' [Objectiven].]

In all directions these same ideals can be seen at work--rigour of proof [streng zu beweisen], precise delimitation of the extent of validity [die Giltigkeitsgrenzen genau zu ziehen], and as a means to this, sharp definition of concepts [die Begriffe sharf zu fassen]. [section 1, page 1]

Proceeding along these lines, we are bound eventually to come to the concept of Number [Begriff der Anzahl], and to the simplest proposition holding of positive whole numbers, which form the foundations of the whole of arithmetic. ...it is in the nature of mathematics always to prefer proof, where proof is possible, to any confirmation by induction... The aim of proof is, in fact, not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another...[BB:inferential clarification, not (just) project of certainty] The further we pursue these enquiries, the fewer become the primitive truths [Urwahrheiten] to which we reduce [zurückführen] everything; and this simplification [Vereinfachung] is in itself a goal worth pursuing. But there may even be justification for a further hope: if, by examining the simplest cases, we can bring to light [zum Bewusstsein bringt] what mankind has there done by instinct, and can extract from such procedures what is universally valid [Allgemeingiltige] in them, may we not thus arrive at general methods for forming concepts [Begriffsbildung] and establishing principles which will be applicable also in more complicated cases? [s2,p2]

It not uncommonly happens that we first discover the content of a proposition [den Inhalt eines Satzes gewinnt], and only later give the rigorous proof [strengen Beweis] of it... In general, therefore, the question of how we arrive at the content of a judgement should be kept distinct from the other question, Whence do we derive the justification [Berechtigung] for its assertion [Behauptung]. [3;3]

Now these distinctions between a priori and a posterior, synthetic and analytic, concern, as I see it* [*: By this I do not, of course, mean to assign a new sense to these terms, but only to state accurately what earlier writers, Kant in particular, have meant by them.] not the content of the judgement but the justification for making the judgement. Where there is no such justification, the possibility of drawing the distinctions vanishes... When a proposition is called a posteriori or analytic in my sense, this is not a judgement about the conditions, psychological, physiological and physical, which have made it possible to form the content of the proposition in our consciousness; nor is it a judgement about the way in which some other man has come, perhaps erroneously, to believe it true; rather, it is a judgement about the ultimate ground upon which rests the justification for holding it to be true... The problem becomes...that of finding the proof of the proposition, and of following it right back to the primitive truths. If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one, bearing in mind that we must take account also of all propositions on which the admissibility of any of the definitions depends. If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some special science, then the proposition is a synthetic one. For a truth to be a posteriori, it must be impossible to construct a proof of it without including an appeal to facts, i.e. to truths [BB:facts=truths] which cannot be proved and are not general, since they contain assertions about particular objects. But if, on the contrary, its proof can be derived exclusively from general laws, which themselves nether need nor admit of proof, then the truth is a priori*. [*:If we recognize the existence of general truths at all, we must also admit the existence of primitive laws [Urgesetze], since from mere individual facts nothing follows, unless it be on the strength of a law...] [s3;p3-4] [BB: Here Frege apparently commits himself to formalism about goodnesses of inference, as well as to the Rationalism that sees a principle behind every propriety of practice. But perhaps what he is committed to is only that every such propriety can be codified in a law.]

Starting from these philosophical questions, we are led to formulate the same demand as that which had arisen independently in the sphere of mathematics, namely that the fundamental propositions of arithmetic should be proved, if in any way possible, with the utmost rigour; for only if every gap in the chain of deduction [Schlusskette] is eliminated with the greatest care can we say with certainty on what primitive truths the proof depends; and only when these are known shall we be able to answer our original question. If we now try to meet this demand, we very soon come to propositions which cannot be proved so long as we do not succeed in analyzing concepts which occur in them into simpler concepts or in reducing them to something of greater generality. Now here it is above all Number which has to be either defined or recognized as indefinable. This is the point which the present work is meant to settle*. [*: In what follows, therefore, unless special notice is given, the only 'numbers' under discussion are the positive whole numbers, which give the answer to the question "How many?". [s4;p4-5]

We must distinguish the numerical formulae, such as 2+3=5, which deal with particular numbers, from general laws, which hold good for all whole numbers. [5;5]

I do not see how a number like 437986 could be given to us more aptly than the way Leibniz does it. Even without having any idea of it, we get it by the means at our disposal [in unser Gewalt] nonetheless. Through such definitions we reduce the whole infinite set of numbers to the number one and increase by one, and every one of the infinitely many numerical formulae can be proved from a few general propositions. [s6;p8]

What a pity Mill did not also illustrate the physical facts underlying the numbers 0 and 1! [s7;p9]

If we call a proposition empirical on the ground that we must have made observations in order to have become conscious of its content, then we are not using the word 'empirical' in the sense in which it is opposed to 'a priori'. We are making a psychological statement, which concerns solely the content of the proposition; the question of its truth is not touched. In this sense, all Münchausen's tales are empirical too... [s8;p12]

Leibniz holds...that the necessary truths, such as are found in arithmetic, must have principles whose proof does not depend on examples and therefore not on the evidence of the senses, though doubtless without the senses it would have occurred to no one to think of them. "The whole of arithmetic is innate and is in virtual fashion in us." What he means by the term 'innate' [eingeboren] is explained by another passage, where he denies "that everything we learn is not innate. The truths of number are in us and yet we still learn them, whether it be by drawing them forth from their source when learning them by demonstration (which shows them to be innate), or whether it be..." [s11;p17] [BB: note potentially inferential sense of 'contains' [in uns]].

If we now bring in the other antithesis between analytic and synthetic, there result four possible combinations, of which however one, viz analytic a priori can be eliminated. Those who have decided with Mill in favor of the a posteriori have therefore no second choice, so that there remain only two possibilities for us still to examine, viz. synthetic a priori and analytic. Kant declares for the former. In that case, there is no alternative but to invoke a pure intuition as the ultimate ground of our knowledge of such judgements, hard though it is to say whether this is spatial or temporal, or whatever else it may be. [s12;p17-18]

I cannot even allow an intuition of 100,000, far less of number in general, not to mention magnitude in general. We are all too ready to invoke inner intuition, whenever we cannot produce any other ground of knowledge [German just says 'Grund']. But we have no business in doing so, to lose sight altogether of the word 'intuition'. Kant in his Logic defines it as follows: "An intuition is an individual idea (repraesentatio singularis), a concept is a general idea (repraesentatio per notas communes) or an idea of reflexion (repraesentatio discursiva)."... It follows that the sense of the word 'intuition' is wider in t he Logic than in the Transcendental Aesthetic. In the sense of the Logic we might perhaps be able to call 100,000 an intuition; for it is not a general concept anyhow. But an intuition in this sense cannot serve as the ground of our knowledge of the laws of arithmetic. [s12;p19]

We shall do well in general not to overestimate the extent to which arithmetic is akin to geometry [s13;p19]

For purposes of conceptual thought [das begriffliche Denken] we can always assume the contrary of some one or other of the geometrical axioms, without involving ourselves in any self-contradictions when we proceed to our deductions [Schlussfolgerungen], despite the conflict between our assumptions and our intuition. The fact that this is possible shows that the axioms of geometry are independent of one another and of the primitive laws of logic, and consequently are synthetic. Can the same be said of the fundamental propositions of the science of numbers? Here, we have only to try denying any one of them, and complete confusion [Verwirrung] ensues. Even to think at all seems no longer possible. The basis of arithmetic lies deeper, it seems, than that of any of the empirical sciences, and even than that of geometry. The truths of arithmetic govern all that is numerable [beherrschen das Gebiet des Zählbaren]. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable [alles Denkbare]. Should not the laws of number, then, be connected very intimately with the laws of thought? [s14;p21]

[H]ow do the empty forms [die leeren Formen] of logic come to disgorge so rich a content? [aus sich heraus solchen Inhalt zu gewinnen] [s16/p22]

[I]t is possible for a mathematician to perform quite lengthy calculations without understanding by his symbols anything intuitable, or with which we could be sensibly acquainted [etwas sinnlich Wahrnehmbares]. And that does not mean that the symbols [Zeichen] have no sense [sinnlos]; we still distinguish between the symbols themselves and their content, even though it may be that the content can only be grasped by their aid. We realize perfectly well that other symbols might have been arranged to stand for the same things. All we need to know is how to handle logically the content as made sensible in the symbols and, if we wish to make application to physics, how to effect the transition to the phenomena. It is, however, a mistake to see in such applications the real sense of the propositions; in any application a large part of their generality is always lost, and a particular element enters in, which in other applications is replaced by other particular elements. [s16;p22-23]

However much we may disparage deduction, it cannot be denied that the laws established [begründet] by induction are not enough. New propositions must be derived from them which are not contained [enthalten] in any one of them by itself. No doubt these propositions are in a way contained covertly in the whole set taken together, but this does not absolve us from the labor of actually extracting them [entwickeln] and setting them out in their own right [und für sich herauszustellen].

Instead of linking our chain of deductions [Schlussreihe] direct to any matter of fact, we can leave the fact where it is, while adopting its content in the form of a condition. By substituting in this way conditions for facts throughout the whole chain of reasoning, we shall finally reduce it to a form in which a certain result is made dependent on a certain series of conditions. This truth would be established by thought alone, or, to use Mill's expression, by an artful manipulation of language. It is not impossible that the laws of number are of this type. This would make them analytic judgements, despite the fact that they would not normally by discovered by thought alone; for we are concerned here not with the way in which they are discovered but with the kind of ground on which their proof rests; or in Leibniz's words, "the question here is not one of the history of our discoveries, which is different in different men, but of the connexion and natural order of truths, with is always the same." [Nouveaux Essais]... The truths of arithmetic would then be related to those of logic in much the same way as the theorems of geometry to the axioms. Each one would contain concentrated within it a whole series of deductions [eine ganze Schlussreihe] for future use, and the use of it would be that we need no longer make the deductions [Schl□sse] one by one, but can express [aussprechen] simultaneously the result of the whole series. [s17;p23-24]

On turning now to consider the primary objects [ursprünglich Gegenständen] of arithmetic, we must distinguish between the individual numbers 3,4, and so on, and the general concept of Number. Now we have already decided in favour of the view that the individual numbers are best derived in the way proposed by Leibniz, Mill, Grassman and other, from t he number one together with increase by one, but that these definitions remain incomplete so long as the number one and increase by one are themselves undefined. And we have seen that we have need of general propositions if we are to derive the numerical formulae from these definitions [BB: e.g. associativity]. Such laws cannot, just because of their generality, follow from the definitions of individual numbers, but only from the general concept of Number. It is this concept we shall now submit to closer examination...[s18;p25]

Yet surely we are entitled to demand of arithmetic that its numbers should be adapted for use in every application made of number, even although that application is not itself the business of number. [s19;p26]

The first question to be faced, then, is whether number is definable. [s20;p26]

It is quite true that while I am not in a position, simply by thinking of it differently, to alte the color or hardness of a thing in the slightest, I am able to think of the Iliad either as one poem, or as 24 books, or as some large number of verses... If I give someone a stone with the words: Find the weight of this, I have given him precisely the object he is to investigate [dem ganzen Gegenstand seiner Untersuchung].   But if I place a pile of playing cards in his hands with the words: Find the Number of these, this does not tell him whether I wish to know the number of cards, or of complete packs of cards, or even say of honour cards at skat. To have given him the pile in his hands is not yet to have given him completely the object he is to investigate; I must add some further word--cards, or packs, or honours... The number 1, on the other hand, or 100 or any other Number, cannot be said to belong to the pile of playing cards in its own right, but at most to belong to it in view of the way in which we have chosen to regard it...[s22;p28-29]

When we see a blue surface, we have an impression of a unique sort, which corresponds to the word 'blue'; this impression we recognize again [erkenner wir wieder] when we catch sight of a blue surface. In order to suppose that there is in the same way, when we look at a triangle, something sensible corresponding to the word 'three', we should have to commit ourselves to finding that same thing again in three concepts too; so that something non-sensible would have in it something sensible...The three in it [a triangle] we do not see directly; rather, we see something upon which can fasten an intellectual activity [geistige Thätigkeit] of ours leading to a judgement in which the number 3 occurs. [s24;p32]

One pair of boots may be the same visible and tangible phenomenon as two boots. Here we have a difference in number to which no physical difference corresponds; for two and one pair are by no means the same thing...[s25;p33]

To quote Berkeley: "It ought to be considered that number...is nothing fixed and settled, really existing in things themselves. It is entirely the creature of the mind, considering, either an idea by itself, or any combination of ideas to which it gives one name, and so makes it pass for a unit. According as the mind variously combines its ideas, the unit varies; and as the unit, so the number, which is only a collection of units, doth also vary.... This line of thought may easily lead us to regard number as something subjective. [s25-26;p33]

For number is no whit more an object of psychology or a product of mental processes than, let us say, the North Sea is. The objectivity [Objectivit„t] of the North Sea is not affected by the fact that it is a matter of our arbitrary choice which part of the water on the Earth's surface we mark off and elect to call the 'North Sea'... The botanist means to assert something just as factual when he gives the Number of a flower's petals as when he gives their color. The one depends on our arbitrary choice [Willk□r] just as little as the other. There does, therefore, exist a certain similarity between Number and color; it consists, however, not in our becoming acquainted with them both in external things through the senses, but in their both being objective. [s26;p34]

I distinguish what I call objective from what is handleable or spatial or actual. The axis of the Earth is objective, so is the centre of mass of the solar system, but I should not call them actual [wirklich] in the way the earth itself is so... [In space] there is something objective all the same; everyone recognizes the same geometrical axioms, even if only by his behavior, and must do so if he is to find his way around the world. What is objective in it is what is subject to laws, what can be conceived and judged, what is expressible in words. [emphasis added, s26;p35]

...let us suppose two rational beings such that projective properties and relations are all they can intuit...; and let what the one intuits as a plane appear to the other as a point, and vice versa...In these circumstances they could understand one another quite well and would never realize the difference between their intuitions, since in projective geometry every proposition has its dual counterpart...Over all geometrical theorems they would be in complete agreement, only interpreting the words differently in terms of their respective intuitions. With the word 'point', for example, one would connect one intuition and the other another. We can therefore still say that this word for them has objective meaning [ihnen das Wort etwas Objectives bedeute], provided only that by this meaning we do not understand any of the peculiarities of their respective intuitions. And in this sense the axis of the Earth too is objective. [s26;p36]

It is in this way that I understand objective to mean what is independent of our sensation, intuition, and imagination, and of all construction of mental pictures out of memories of earlier sensations, but not what is independent of reason,--for what are things independent of reason? To answer that would be as much as to judge without judging...[s26;p36]

Now objectivity cannot, of course, be based on any sense-impression, which as an affection of our mind is entirely subjective, but only, so far as I can see, on the reason. [s27;p38]

Some writers define Number as a set or multitude or plurality. All these views suffer from the drawback that the concept will not then cover the numbers 0 and 1. [s28;p38]

Does the number word [Zahlwort] 'one' stand for a property of objects?... It must strike us immediately as remarkable that every single thing should possess this property. It would be incomprehensible why we should still ascribe it expressly to a thing at all. It is only in virtue of the possibility of something not being wise that it makes sense to say "Solon is wise." The content of a concept diminishes as its extension [Umfang] increases; if its extension becomes all-embracing, its content must vanish altogether... In isolation it seems that 'one' cannot be a predicate [Praedicat]. This is even clearer if we take the plural. Whereas we can combine "Solon was wise" and "Thales was wise" into "Solon and Thales were wise", we cannot say "Solon and Thales were one". But it is hard to see why this should be impossible, if 'one' were a property both of Solon and of Thales in the same way that 'wise' is. [s29;p40-41]

...gives as a criterion of a strict unit that it must be incapable of dissection. Obviously, by tightening up its internal cohesion without limit, they hope to arrive at a criterion for their unit which is independent of any arbitrary way of regarding things. This attempt collapses because we are then left with practically nothing fit to be called a unit and numbered. The result is that we at once begin to retrace our steps, by giving as the criterion not that the thing itself should be incapable of dissection in fact, but that we should think of it as such. This brings us back once again to our way of regarding things, with all its fluctuations. [s33;p43]

Every attempt to define 'one' as a property having thus failed, we must finally abandon the view that in designating a thing a unit we are adding to our description of it [eine nähere Bestimmung zu sehen]... Why do we ascribe identity to objects that are to be numbered? And is it only ascribed [zugeschriebene] to them, or are they really identical? In any case, no two objects are ever completely identical. On the other hand, of course, we can practically always engineer some respect in which any two objects whatever agree. And with this we are back once more to our arbitrary way of regarding things [willkürlichen Auffassung]... [s34;p44]

The symbols 1', 1'', 1''' tell the tale of our embarrassment. We must have identity--hence the 1; but we must have difference--hence the strokes; only unfortunately, the latter undo the work of the former... It follows, therefore, that on his view there would not only be distinct ones but also distinct twos and so on; for 1''''+1''''' could not be substituted for [vertreten] 1''+1'''. [s36,38;p48-49]

Only concept words [Begriffsw”rten] can form a plural. [s38;p50]

We are faced, therefore, with the following difficulty: If we try to produce the number by putting together distinct objects, the result is an agglomeration in which the objects contained remain still in possession of precisely those properties which serve to distinguish them from one another; and that is not the number. But if we try to do it the other way, by putting together identicals, the result runs perpetually together into one and we never reach a plurality. If we use 1 to stand for [bezeichnen] each of the objects being numbered, we make the mistake of assigning the same symbol [Zeichen] to different things. But if we provide the 1 with differentiating strokes, it becomes unusable for arithmetic. [s39;p50]

It is no good objecting that 0 and 1 are not numbers in the same sense as 2 and 3. What answers the question "How many?" is number, and if we ask, for example, "How many moons has this planet?", we are quite as much prepared for the answer 0 or 1 as for 2 or 3, and that without having to understand the question differently. [s44;p57]

Number is not abstracted from things in the way in that color, weight, and hardness are, nor is it a property of things in the sense that they are. But when we make a statement of number, what is that of which we assert something? [von wem durch eine Zahlangabe etwas asugesagt werde] [s45;p58]

How are we to curb the arbitrariness of our ways of regarding things [die Willkühr der Auffassung], which threatens to obliterate every distinction between one and many? [45]

...consider number in the context of a judgement which brings out its basic use...

...objects too can change their properties without that preventing us from recognizing them as the same. [sie als dieselben anzuerkennen] [46]

It is impossible to speak of an object without in some way designating or naming it. (sec. 47)

But the word "whale" is not the name of any individual creature. [Das Wort "Wallfisch" benennt aber kein Einzelwesen.] If it be replied that what we are speaking of is not, indeed, an individual definite object [von einem einzelnen, bestimmten Geganstande die Rede], but nevertheless an indefinite object, I suspect that "indefinite object" is only another term for a concept, and a poor one at that, being self-contradictory. [47]

However true it may be that our proposition can only be verified by observing particular animals, that proves nothing as to its content; to decide what it is about [wovon es handelt] we do not need to know whether it is true or not, nor for what reasons we believe it to be true. [47]

Here he [Spinoza] makes the mistake of supposing that a concept can only be acquired by direct abstraction from a number of objects. We can, on the contrary, arrive at a concept equally well by starting from defining characteristics; and in such a case it is possible for nothing to fall under it. If this did not happen, we should never be able to deny existence, and so the assertion of existence too would lose all content. [49]

It will not do to call a general concept word [Begriffswort] the name of a thing....Only when conjoined with the definite article or a demonstrative pronoun can it be counted as the proper name of a thing, but in that case it ceases to count as a concept word. The name of a thing is a proper name. ...With a concept the question is always whether anything, and if so what, falls under it. With a proper name such questions make no sense...As soon as a word is used with the indefinite article or in the plural without any article, it is a concept word. [51]

By properties which are asserted of a concept I naturally do not mean the characteristics which make up the concept. These latter are properties of things which fall under the concept [die unter den Begriff fallen], not of the concept...In this respect existence is analogous to number. Affirmation of existence is in fact nothing but denial of the number nought. Because existence is a property of concepts, the ontological argument for the existence of God breaks down.... However, it would be wrong to conclude that it is in principle impossible ever to deduce from a concept, that is, from its component characteristics, anything which is a property of the concept. [BB: F is leaving room for 'logical objects']... In this way we can make one concept fall under another higher or, so to say, second order concept. This relationship, however, should not be confused with the subordination of species to genus. [53]

Why not, in fact, adopt this very apt suggestion, and call a concept the unit relative to the Number which belongs to it? [...Einheit zu nennen in Bezug auf die Anzahl welche ihm zukommt] [54]

The concept "syllables in the word 'three'" picks out the word as a whole, and as indivisible in the sense that no part of it falls any longer under the same concept. Not all concepts possess this quality. We can, for example, divide up something falling under the concept "red" into parts in a variety of ways, without the parts thereby ceasing to fall under the same concept "red". To a concept of this kind no finite number will belong. The proposition asserting that units are isolated and indivisible can, accordingly, be formulated as follows: **Only a concept which isolates what falls under it in a definite manner, and which does not permit any arbitrary division of it into parts, can be a unit** relative to a finite Number. [54]

Now that we have learned **that the content of a statement of number is an assertion about a concept**...[55]

Moreover, we cannot by the aid of our suggested definitions prove that, if the number a belongs to a concept F and the number b belongs to the same concept, then necessarily a=b. Thus we should be unable to justify the expression "the number which belongs to the concept F", and therefore should find it impossible in general to prove a numerical identity, since we should be quite unable to achieve a determinate number. It is only an illusion that we have defined 0 and 1, in reality we have only fixed the senses of the expressions: "the number 0 belongs to" "the number 1 belongs to"; **but we have no authority to pick out the 0 and 1 here as self-subsistent object that can be recognized as the same again**. [selfständige, wiedererkennbare Gegenstände zu unterscheiden] [56]

In the proposition "the number 0 belongs to the concept F", 0 is only an element in the predicate (taking the concept F to be the real subject). For this reason I have avoided calling a number such as 0 or 1 or 2 a property of a concept. Precisely because it forms only an element in what is asserted, the individual number shows itself for what it is, a self-subsistent object. I have already drawn attention to the fact that we speak of "the number 1", where **the definite article serves to class it as an objec**t.[57]

For example, the proposition "Jupiter has four moons" can be converted [umsetzen] into "the number of Jupiter's moons is four."... So what we have is an identity, stating that the expression "the number of Jupiter's moons" signifies the same object as the word "four". [57]

It may be that every word calls up some sort of idea in us, even a word like 'only'; but this idea need not correspond to the content of the word; it may be quite different in different men. The sort of thing we do is to imagine a situation where some proposition in which the word occurs would called for...[59]
[BB: Wittgenstein must have meditated long and hard about this passage.]

Time and time again **we are led by our *thought* beyond the scope of our *imagination*, without thereby forfeiting the support we need for our *inferences***...[BB: my italics] That we can form no idea of its content is therefore no reason for denying all meaning [Bedeutung] to a word... But we ought always to keep before our eyes a complete proposition. **Only in a proposition have the words really a meaning...It is enough if the proposition taken as a whole has a sense; it is this that confers [erhalten] on its parts also their content.** [sec. 60]

This observation [that it is enough if proposition as a whole have sense] is destined, I believe, to throw light on quite a number of difficult concepts, among them that of the infinitesimal*, and its scope is not restricted to mathematics either. {*: The problem here is not, as might be thought, to produce a segment bounded by two distinct points whose length is dx, but rather to define the sense of an identity of the type: df(x)=g(x)dx.} [60]
[BB: This is an illuminating example. The alternative rejected in the note is to produce what is referred to by an infinitesimal-expression. Infinitesimals in fact were hypostatized from equations of just this sort, resulting from a transform of df(x)/dx=g(x). F wants us to focus on the recognition statement.]

The self-subsistence [Selbständigkeit] which I am claiming for number is not to be taken to mean that a number word signifies something when removed from the context of a proposition, but only to preclude the use of such words as predicates or attributes, which appreciably alters their meaning. [60]

Yet even granted that what is subjective has no position in space, how is it possible for the number 4, which is objective, not to be anywhere? Now I contend that there is no contradiction in this whatever. It is a fact that the number 4 is exactly the same for everyone who deals with it; but that has nothing to do with being spatial. Not every objective object [objectives Gegenstand] has a place. [61]

To obtain the concept of Number, one must fix [feststellen] **the sense of a numerical identity** [Zahlengleichung].

How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them? [Note: this is an epistemic question, and denying the possibility of Anschauungen is meant to deny causal intercourse.] Since it is only in the context of a proposition that words have any meaning, **our problem becomes this: To define the sense of a proposition in which a number word appears.** That, obviously, leaves us still a very wide choice. But we have already settled [festgestellt] that number words are to be understood as standing for self-subsistent objects. [dass unter den Zahlwoerten selbständige Gegenstände zu verstehen sind-- that by number words, self-subsistent objects should be understood] And that is enough to give us a class of propositions which must have a sense, namely those which express our recognition of a number as the same again. [der Sätze, welche ein Wiedererkennen ausdrücken.] If we are to use a symbol a to signify [bezeichnen] an object, we must have a criterion for deciding in all cases whether b is the same as a, even if it is not always in our power to apply this criterion.
[BB: This actually *says* only that recognition judgements having senses is *necessary* for the introduction of singular terms and hence objects. But he goes on to treat it as *sufficient*.]

In our present case, we have to define the sense [Sinn] of the proposition the number which belongs to the concept F is the same as that which belongs to the concept G that is to say, we must reproduce the content [Inhalt] of this proposition in other terms, avoiding the use of the expression the Number which belongs to the concept F In doing this, we shall be **giving a general criterion for the identity of numbers** [Kennzeichen für die Gleichheit von Zahlen]. **When we have thus acquired a means of [a] arriving at a determinate number and of [b] recognizing it again as the same, we can assign it a number word as its proper name** [zum Eigennamen geben]. [62]
[BB: Use of 'Inhalt' here is important: we know what that means from BGS, and it is an inferential notion. It is interesting that he commits himself here to: if t is a name, then identities must have a sense for it, but then draws conclusions that would follow from the converse. In fact, then, he is treating playing a role in identities as nec and suf for being a singular term. It is interesting also to see why this paragraph counts as answering the question with which it begins.]

This opinion, that numerical equality or identity must be defined in terms of one-to-one correlation...raises certain logical doubts and difficulties... It is not only among numbers that the relationship of identity is found. From which it seems to follow that we ought not to define it specially for the case of numbers. We should expect the concept of identity to have been fixed first, and that then, for it together with the concept of Number, it must be possible to deduce when Numbers are identical with one another, without there being any need for this purpose of a special definition of numerical identity as well.

As against this, it must be noted that for us the concept of Number has not yet been fixed, but is only due to be determined in the light of our definition of numerical identity [sondern erst mittels unserer Erklärung bestimmt werden soll**.]** **Our aim is to construct the content of a judgement** [den Inhalt eines Urtheils zu bilden] **which can be taken as** [auffassen läst] **an identity such that each side of it is a number**. We are therefore proposing not to define identity specially for this case, but to use the concept of identity, taken as already known [des schon bekannten Begriffes der Gleichheit], as a means for arriving at that which is to be regarded as being identical. [63]
[BB: Notice that in first sentence, it is the circumstances, not the consequences of application that are specified. Cons of app are settled by its being an identity statement.]

The judgement "line a is parallel to line b", or using symbols a//b, can be taken as an identity [als Gleichung aufgeffasst werden]. If we do this, we obtain the concept of direction, and say: "the direction of line a is identical with the direction of line b". Thus we replace the symbol // by the more generic symbol =, through removing what is specific in the content of the former and dividing it between a and b. **We carve up the content in a new way from the original way, and this yields us a new concept.** Often, of course, we conceive the matter the other way around, and many authorities define parallel lines as lines whose directions are equal... ...[This] is only made possible by surreptitiously assuming, in our use [Gebrauch] of the word 'direction' what was to be proved**: for if it were false that "straight lines parallel to the same straight line are parallel to one another", then we could not transform** [verwandeln] **a//b into an identity.** [64]
[BB: This 'carving' to get 'new concepts' is always done substitutionally.]

Now in order to get, for example, from parallelism* [*:...The argument can readily be transferred in essentials to apply to the case of numerical identity.] to the concept of direction, let us try the following definition: The proposition "line a is parallel to line b" is to mean the same as "the direction of line a is identical with the direction of line b".

...gives rise to a second doubt--are we not liable, through using such methods, to become involved in conflict with the well-known laws of identity? Let us see what these are. As analytic truths they should be capable of being derived from the concept itself alone. Now Leibniz's definition is as follows: "Things are the same as each other, of which one can be substituted for the other without loss of truth." [Eadem sunt, quorum unum potest substitui alteri salva veritate.] This I propose to adopt as my own definition of identity. ...Now it is actually the case that **in universal substitutability [allgemeinen Ersetzbarkeit] all the laws of identity are contained**.
[BB: This last sentence explains why and *how* identity claims have their inferential content determined substitutionally.]

In order, therefore, to justify [rechtfertigen] our proposed definition of the direction of a line, we should have to show that it is possible, if line a is parallel to line b, to substitute

"the direction of b" everywhere for "the direction of a". This task is made simpler by the fact that we are being taken initially to know of nothing that can be asserted about the direction of a line except the one thing, that it coincides with the direction of some other line. We should thus have to show only that substitution was possible in an identity of this type, or in judgement-contents containing such identities as constituent elements* [in Inhalten welche solche Gleichheiten als Bestandteile enthalten würden] [*:In a hypothetical judgement, for example, an identity of directions might occur as an antecedent or consequent.] The meaning of any other type of assertion about directions would have first of all to be defined, and in defining it we can make it a rule always to see that it must remain possible to substitute for the direction of any line the direction of any line parallel to it. [65]
[BB:This footnote expresses Frege's concern with the embedded use of concepts as an equal explanatory target with free-standing uses. This is the basis of his concern with compositionality (or decompositionality). The method of procedure he describes here is just that he will follow in the GG.]

But there is still a third doubt which may make us suspicious of our proposed definition. In the proposition "the direction of a is identical with the direction of b" the direction of a plays the part of [erscheint als] an object,* and our definition affords us a means of recognizing this object as the same again [wir haben in unserer Definition ein Mittel, diesen Gegenstand wiederzuerkennen] in case it should happen to crop up in some other guise [Verkleidung], say as the direction of b.
[*:This is shown by the definite article. A concept is for me that which can be predicate of a singular judgement-content [ein mögliches Prädicat eines singulären beurtheilbaren Inhalts], an object that which can be subject of the same. If in the proposition "the direction of the axis of the telescope is identical with the direction of the Earth's axis" we take the direction of the axis of the telescope as subject, then the predicate is "identical with the direction of the Earth's axis". This is a concept. But the direction of the Earth's axis is only an element in the predicate; it, since it can also be made the subject, is an object.]

But this means does not provide for all the cases. **It will not, for instance, determine whether England is the same as the direction of the Earth's axis**...Our definition of direction...says nothing as to whether the proposition "the direction of a is identical with q" should be affirmed or denied, except for the one case where q is given in the form "the direction of b". What we lack is the concept of direction; for if we had that, then we could lay it down that, if q is not a direction, our proposition is to be denied, while if it is a direction, our original definition will decide whether it is to be denied or affirmed...But then we have obviously come round in a circle. For in order to make use of this definition, we should have to know already in every case whether the proposition "q is identical with the direction of b" was to be affirmed or denied. [66]

If we were to try saying "q is a direction if it is introduced [eingeführt] by means of the definition set out above, then we should be treating the way in which the object q is introduced as a property of q, which it is not. The definition of an object does not, as such, really assert anything about the object, but only lays down the meaning of a

symbol. After this has been done, the definition transforms itself into a judgement, which does assert about the object; but now it no longer introduces the object, it is exactly on a level with other assertions made about it. If, moreover, we were to adopt this way out, **we should have to be presupposing that an object can only be given in one single way** [dass ein Gegenstand nur auf eine einzige Weise gegeben werden könnte]; for otherwise it would not follow, from the fact that q was not introduced by means of our definition, that it could not have been introduced by means of it. All identities would then amount simply to this, that whatever is given to us in the same way is to be reckoned as the same. This, however, is a principle so obvious and sterile [so selbstverständlich und so unfruchtbar] as not to be worth stating. **We could not, in fact, draw from it any conclusion which was not the same as one of our premises. Why is it after all that we are able to make use of identities with such significant results in such diverse fields? Surely it is rather because we are able to recognize something as the same again even though it is given in a different way** [dass man etwas wiedererkennen kann, obwohl es auf verscheidene Weise gegeben ist]. [67]
[BB: This passage should be compared very carefully with the first few paragraphs of USB from 7 years later. Fruitfulness, in terms of inferential licensing of novel assertional commitments, is already the key (a Kantian idea, see the Logik.]

If a waiter wishes to be certain of laying exactly as many knives on a table as plates, he has no need to count either of them; all he has to do is to lay immediately to the right of every plate a knife, taking care that every knife on the table lies immediately to the right of a plate. Plates and knives are thus correlated one to one, and that by the identical spatial relationship. Now if in the proposition [Satze] "a lies immediately to the right of A" we conceive first one and then another object inserted in place of a and again of A, then that part of the content which remains unaltered throughout this process constitutes the essence of the relation [für a und A andere und andere Gegenstände eingesetzt denken, so macht der hierbei unverändert bleibende Theil des Inhalts das Wesen der Bezeihung aus]. What we need is a generalization of this. [BB: Note that so far what is at issue is spatial relations, not anything to do with semantic contents. But 'Satze' is used both here and below. Is it expressions or objects (or both) that are substituted below?]

If from a judgement-content which deals with an object a and an object b we subtract a and b, we obtain as remainder a relation-concept which is, accordingly, incomplete at two points [Indem wir von einem beurtheilbaren Inhalte, der von einem Gegenstande a und von einem Gegenstande b handelt, a und b absondern, so behalten wir einen Beziehungsbegriff übrig, der demnach in doppelter Weise ergänzungsbedürftig ist].[BB: So *concepts* are extractable from judgeable *contents*.] If from the proposition "the Earth is more massive than the Moon" we subtract [separate] "the Earth", we obtain the concept "more massive than the Moon". If, alternatively, we subtract the object, "the Moon", we get the concept "less massive than the Earth". But if we subtract them both at once, then we are left with a relation-concept, which taken by itself has no [assertible] sense any more than a simple concept has: it has always to be completed in order to make up a judgement-content [Sondern wir beide zugleich ab, so belieft ein Beziehungsbegriff zurück, der für sich allein ebensowenig wie ein einfacher Begriff einen Sinn hat: er verlangt immer eine Ergänzung zu einem beurteilbaren Inhalte]. It can however be

complete in different ways: instead of Earth and Moon I can put, for example, Sun and Earth, and this eo ipso effects the subtraction [Aber diese kann in verscheidener Weise geschehen: satt Erde und Mond kann ich z.B. Sonne und Erde Setzen, und hierdurch wird eben die Absonderung bewirkt]. [70]
[BB: This last sentence makes it perfectly clear that the talk of Absonderung or separation (isolation) is metaphorical for substitutional talk.]

The study of relation-concepts belongs, like that of simple concepts, to pure logic [Der Beziehungsbegriff gehört also wie der einfache der reinen Logik an]. What is of concern to pure logic is not the special content of any particular relation, but only the logical form. And whatever can be asserted of this, is true analytically known a priori. This is as true of relation-concepts as of other concepts. Just as "a falls under the concept F" is the general form of a judgement-content which deals with an object a, so we can take "a stands in the relation R to b" as the general form of a judgement-content which deals with an object a and an object b. [70]

It may still be asked, what is the meaning of the expression "every object which falls under F stands in the relation R to an object falling under G" in the case where no object at all falls under F. I understand this expression as follows: the two propositions "a falls under F" and "a does not stand in the relation R to any object falling under G" cannot, whatever be signified by a, both be true together: so that either the first proposition is false, or the second is, or both are [können nicht mit einander bestehen, was auch a bezeichne, sodass entweder der erste oder der zweite oder beide falsch sind {doesn't use 'wahr', here or in unquoted remainder of section.}]. [71]
[BB: Note that this essential quantificational idiom is explained in terms of a) substitution instance and b) incompatibility of two claims, so that it is inappropriate to maintain (=be assertionally committed to) them simultaneously.]

[Concepts that contain contradictions]...cannot do any harm, if only we do not assume that there is anything which falls under them--and to that we are not committed by merely using them [und das thut man durch den blossen Begrauch der Begriffe noch nicht]. That a concept contains [enthalte] a contradiction is not always obvious [offensichtlich] without investigation; but to investigate it we must first **possess** it and, in logic, treat it just like any other. **All that can be demanded of a concept from the point of view of logic and with an eye to rigour of proof is only that the limits to its application should be sharp, and that it should be determined definitely about every object whether it falls under that concept or not.** But this demand is completely satisfied by concepts which, like "not identical with itself", contains a contradiction: for of every object we know that it does not fall under any such concept.*

[*: The definition of an object [Definition eines Gegenstandes {peculiar phrase—Isn't it expressions that are usually defined? or does this mean 'picked out' or making definite, moving towards the body-builders' use of 'definition'? cf 'bestimmte' below}] in terms of a concept under which it falls is a very different matter. For example, the expression "the largest proper fraction" has no content, since the definite article purports to refer to a definite object [der bestimmte Artikel den Anspruch erhebt, auf einen bestimmten

Gegenstand hinzuweisen]. On the other hand, the concept "fraction smaller than 1 and such that no fraction smaller than one exceeds it in magnitude" is quite unexceptionable: in order, indeed, to prove that there exists no such fraction, we must make use of just this concept, despite its containing a contradiction**. If, however, we wished to use this concept for defining an object falling under it, it would, of course, be necessary first to show two distinct things: 1. that some object falls under this concept; 2. that only one object falls under it.** Now since the first of these propositions, not to mention the second, is false, it follows that the expression "the largest proper fraction" is senseless. {This is the origin of Russell's theory of definite descriptions}]

On my use of the word 'concept', "a falls under the concept F" is the general form of a judgement-content [Form eines beurtheilbaren Inhalts ist] which deals with an object a and remains judgeable when anything else replaces a [von einem Gegenstande a handelt und der beurtheilbar bleibt, was man auch für a setze {Austin has: "permits of the insertion for a of anything whatever"}]. And in this sense "a falls under the concept 'not identical with itself'" has the same meaning as "a is not identical with itself" or "a is not identical with a".
[BB: Notice in the first paragraph that this definiteness of application criterion is forwarded only as appropriate in the context of an inquiry where the rigor of proof is a primary aim. Also, this first paragraph is already an important text for contextually defining "object falling under a concept"]

...to use the symbol '=' is likewise **to designate** [something] **an object.** [76]

So regarded, our Number [aleph one] has a character as definite as that of any finite Number; it can be recognized again beyond doubt as the same, and can be distinguished from every other. [84]

...in the external world, in the whole of space and all that therein is, there are no concepts, no properties of concepts, no numbers. **The laws of number, therefore, are not really applicable to external things; they are not laws of nature. They are, however, applicable to judgements holding good of things in the world: they are the laws of the laws of nature. They assert not connections between phenomena, but connections between judgements; and among judgements are included the laws of nature.** [87]

Kant obviously--as a result no doubt of defining them too narrowly, underestimated the value of analytic judgements, though it seems that he did have some inkling of the wider sense in which I have used the term* {* At B14 he says that a synthetic proposition can only be seen to be true by the law of contradiction, if another synthetic proposition is presupposed.} On the basis of his definition, the division of judgements into synthetic and analytic is not exhaustive. What he is thinking of is the simple affirmative judgement; there, we can speak of a subject concept and ask--as his definition requires--whether the predicate concept is contained in it or not. But how can we do this, if the subject is an existential one? In these cases there can simply be no question of a subject concept in Kant's sense.

[BB: The manifest issue here concerns the possibility of analytic existential judgements. Of course, for Kant, existential judgements are one and all synthetic—just because of his reading of the proof structure of Euclid's geometry, as Friedman has shown.]

He seems to think of concepts as defined by giving a simple list of characteristics in no special order; **but of all the ways of forming concepts, that is one of the least fruitful**...If we represent the concepts (or their extensions) by figures or areas in a plane, then the concept defined by a simple list of characteristics corresponds to the area common to all the areas representing the defining characteristics; it is enclosed by segments of their boundary lines. With a definition like this, therefore, what we do...is to use the lines already given in a new way for the purposes of demarcating an area.* {*: Similarly, if the characteristics are joined by "or".} Nothing essentially new, however, emerges in the process. **But the most fruitful kind of** definition [Die fruchtbareren Begriffsbestimmungen] **is a matter of drawing boundary lines that were not previously given at all. What we shall be able to infer from it cannot be inspected** [übersehen] **in advance; here, we are not simply taking out of the box again what we have just put into it. The conclusions we draw from it extend our knowledge** [Diese Folgerungen erweitern unsere Kenntnisse], and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic. The truth is that they are contained in the definitions [in den Definitionen enthalten], but as plants are contained in their seeds, not as beams are contained in a house. **Often we need several definitions for the proof of some proposition, which consequently is not contained in any one of them alone, yet does follow logically from all of them together.** [88]

Moreover the excessive variety of logical forms that has gone into the shaping of our language makes it difficult to isolate a set of modes of inference which is both sufficient to cope with all cases and easy to take in at a glance. To minimize these drawbacks, I invented my concept writing. It is designed to produce expressions which are shorter and easier to take in, and to be operated like a calculus [Art einer Rechnung] by means of a small number of standard moves, so that no step is permitted which does not conform to the rules, which are laid down once and for all.* **{*: It is designed, however, to be capable of expressing not only the logical *form*, like Boole's notation, but also the *content* of a proposition**.{My italics.} It is impossible, therefore, for any premiss to creep into a proof without being noticed. In this way I have, without borrowing any axiom from intuition, given a proof of a proposition which might at first sight be taken for synthetic, which I shall here formulate as follows:
f the relation of every member of a series to its successor is one- or many-one, and if m and y follow in that series after x, then either y comes in that series before m, or it coincides with m, or it follows after m.
From this proof it can be seen that **propositions which extend our knowledge** [welche unsere Kenntnisse erweitern] **can have analytic judgements for their content** [analytische Urtheile enthalten koennen]. [91]
[BB: On judgements that extend our knowledge, see the opening 'graphs of USB]

Strictly, of course, we can only establish that a concept is free from contradiction by first

producing something that falls under it. The converse inference is a fallacy. ...there is something to prevent us from regarding (2-3) without more ado as a symbol which solves the problem; for an empty symbol [leeres Zeichen] is precisely no solution; without some content it is merely ink or print on paper, as which it possesses physical properties but not that of making 2 when increased by 3**. Really it is not a symbol at all, and to use it as one would be a mistake in logic.** Even for c>b, it is not the symbol ("c-b") [BB: why are the quote-marks placed inside the parentheses? Do the latter indicate term formation for F?] that solves the problem, but its content. [95]
[BB: The emphasized sentence picks up the policy evidently in use in BGS, of supposing that the expression by BGS primitives of, for instance, judgeable content, and what it is, has been settled unambiguously *before* one begins to calculate in BGS at all.]

This is the error that infects the formalist theory of fractions and of negative and complex numbers* {*: Cantor's infinite Numbers are in like case.} It is made a postulate that the familiar rules of calculation shall still hold, where possible, for the newly introduced numbers, and from this their general properties and relations are deduced. If no contradiction is anywhere encountered, the introduction of new numbers is held to be justified, as though it were impossible for a contradiction still to be lurking somewhere nevertheless, and **as though freedom from contradiction amounted straight away to existence**. [96]
[BB: What does the distinction between these last two conditions amount to? Perhaps this, that one must be able to specify, in some antecedent vocabulary, perhaps a canonical one, an object identical to one specified in the new vocabulary.]

That this mistake is so easily made is due, of course, to the failure to distinguish clearly between concepts and objects. Nothing prevents us from using the concept "square root of -1"; but we are not entitled to put the definite article in front of it without more ado and take the expression "the square root of -1" as having a sense. [97]
[BB: to turn a predicate into a definite description, to be entitled to prepend the definite article, must prove existence and uniqueness.]

Later, he symbolizes an operation which is thetic, one-valued and associative by (a+b)...An operation which etc.? But which one? Any we care to choose? Then that is not a definition of (a+b); and besides, what if none exists? ...what we cannot say is: we propose to call an operation of this sort the operation of addition, and to symbolize it by (a+b) [that is, by a singular term expression]. For it has not yet been established that there is one and only one such operation. We cannot define by putting on one side of our identity the indefinite article and on the other the definite. [Man darf nicht auf einen Seite einer Definitionsgleichung den unbestimmetn und auf der andern den bestimmten Artikel gebrauchen.] [98]

In a word, this purely formalist theory is not sufficient. What is valuable in it is simply this. We can prove that if any operation possesses certain properties...then certain propositions hold good of it [Sätze von ihnen gelten]. So that if we go on to show that addition and multiplication, which are already known to us [welche man schon kennt], possess these properties, we can then proceed immediately to assert our propositions of addition and multiplication. [99]

We should now be able to prove the formula for cos(n,a) if we knew that from the identity of complex numbers, the identity of their real parts can be inferred... Well, perhaps it is indeed possible to assign a whole variety of different meanings to a+bi, and to sum and product, all of them such that those laws continue to hold good; but **it is not immaterial whether we can or cannot find some such a sense for those expressions**. [101] [BB: What is the test (def., condition, criterion) of doing or having done so? (What does one have to have done, in order to have done this? And what are the rules (including specifications of permissible vocabulary) for specifying the first?]

It is common to proceed as if a mere postulation were equivalent to its own fulfillment. We postulate that it shall be possible in all cases to carry out the operation of subtraction, or of division, or of root extraction, and suppose that with that we have done enough. But why do we not postulate that through any three points it shall be possible to draw a straight line? Why do we not postulate that all the laws of addition and multiplication shall continue to hold for a three-dimensional complex number system just as they do for real numbers? Because this postulate contains a contradiction. Very well then, what we have to do first is to prove that these other postulates of ours do not contain any contradiction.

...it is not immaterial to the cogency of our proof whether "a+bi" has a sense or is nothing more than printer's ink. It will not get us anywhere simply to require that it have a sense, or to say that it is to have the sense of the sum of a and bi, when we have not previously defined what "sum" means in this case and when we have given no justification for the use of the definite article.
[BB: That Rechtfertigung comes in two parts, existence and uniqueness:
show there is some, and show there is no more than one. The first involves, I think, a relation to antecedent or canonical vocabulary. The latter is a matter of establishing substitution properties.]

How are complex numbers to be given to us then, and fractions and irrational numbers? If we turn for assistance to intuition, we import something foreign into arithmetic; but if we only define the concept of such a number by giving its characteristics, if we simply require the number to have certain properties, then there is still no guarantee that anything falls under the concept and answers to our requirements, and yet it is precisely on this that proofs must be based. [104]

In the same way with the definitions of fractions, complex numbers and the rest, **everything will in the end come down to the search for a judgement-content** [beurtheilbaren Inhalt] **which can be transformed** [verwandelt] **into an identity** whose sides precisely are the new numbers. In other words, **what we must do is fix the sense of a recognition-judgement for the case of these numbers.** [104]

For every object there is one type of proposition which must have a sense, namely the recognition-statement, which in the case of numbers is called an identity...The

problem, therefore, was this: to fix the sense of a numerical identity... [106]

When are we entitled to regard a content as that of a recognition-judgement? For this a certain condition has to be satisfied, namely that it must be possible in every judgement to substitute without loss of truth the right-hand side of our putative identity for its left-hand side. [107]

Here, just as there, **it is a matter of fixing the content of a recognition-judgement** [Wiedererkennungsurtheils] [109]