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Kant and the Exact Sciences

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Contents

· CHAPTER 1 ·

Tractatus 4.0412: For the same reason the idealist's appeal to 'spatial spectacles' is inadequate to explain the seeing of spatial relations, because it cannot explain the multiplicity of these relations.

Geometry

Since the important work of early twentieth-century philosophers of geometry such as Russell, Carnap, Schlick and Reichenbach, Kant's critical theory of geometry has not looked very attractive. After their work and the work of Riemann, Hilbert, and Einstein from which they drew their inspiration, Kant's conception is liable to seem quaint at best and silly at worst. His picture of geometry as somehow grounded in our intuition of space and time appears thoroughly wrong; and there is a consequent tendency to view the Transcendental Aesthetic as an unfortunate embarrassment that one has simply to rush through on the way to the more relevant and enduring insights of the Analytic.[1]

The standard modern complaint against Kant runs as follows. Kant fails to make the crucial distinction between *pure* and *applied* geometry. Pure geometry is the study of the formal or logical relations between propositions in a particular axiomatic system, an axiomatic system for Euclidean geometry, say. As such it is indeed a priori and certain (as a priori and certain as logic is, anyway), but it involves no appeal to spatial intuition or any other kind of experience. Applied geometry, on the other hand, concerns the truth or falsity of such a system of axioms under a particular interpretation in the real world. And, in this connection, it matters little whether our axioms are interpreted in the physical world—in terms of light rays, stretched strings, or whatever, or in the psychological realm—in terms of "looks" or "appearances" or other phenomenological entities. In either case the truth (or approximate truth) of any particular axiom system is neither a priori nor certain but, rather, a matter for empirical investigation, in either physics or psychology. This modern attitude is epitomized in Einstein's famous dictum (in which he has geometry

1. One finds this attitude in even as sympathetic and sensitive a commentary as Kemp Smith [57], for example, pp. 40–41.

especially in mind): "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality" ([21], p. 28). From this point of view, then, Kant misconstrues the problem from the very beginning, and, accordingly, his teaching is hopelessly confused.

Yet this modern complaint is quite fundamentally unfair to Kant; for Kant's conception of *logic* is certainly not our modern conception. Our distinction between pure and applied geometry goes hand in hand with our understanding of logic, and this understanding simply did not exist before 1879, when Frege's *Begriffsschrift* appeared. The importance of relating Kant's understanding of logic to his philosophy of mathematics has been stressed by several recent commentators, notably, by Hintikka and Parsons.[2] In reference to geometry in particular, however, I think that no one has been as close to the truth as Russell, who habitually blamed all the traditional obscurities surrounding space and geometry—including Kant's views, of course—on ignorance of the modern theory of relations and uncritical reliance on Aristotelian subject-predicate logic.[3] I think Russell is exactly right, but I would like to turn his polemic on its head. Instead of using our modern conception of logic to disparage and dismiss earlier theories of space, we should use it as a tool for interpreting and explaining these theories, for deepening our understanding of the difficult logical problems with which they were struggling. This, in any case, is what I propose to undertake in reference to Kant's theory in what follows.

· I ·

What is most striking to me about Kant's theory, as it was to Russell, is the claim that geometrical *reasoning* cannot proceed "analytically according to concepts"—that is, purely logically—but requires a further activity called "construction in pure intuition." The claim is expressed most clearly in the Discipline of Pure Reason in Its Dogmatic Employment, where Kant contrasts philosophical with mathematical reasoning:

> Philosophy confines itself to general concepts; mathematics can achieve nothing by concepts alone but hastens at once to intuition, in which it considers

2. For Hintikka see [50]; "Kant's 'New Method of Thought' and His Theory of Mathematics" (1965) and "Kant on the Mathematical Method" (1967), both reprinted in [53]; and [52]. For Parsons see "Infinity and Kant's Conception of the 'Possibility of Experience' " (1964) and "Kant's Philosophy of Arithmetic" (1969), both reprinted in [92]; [90]; and [91]. See, in addition, Beth [6], which inspired Hintikka, and Thompson [108]. This last is oriented around the role of intuition in empirical knowledge, but it also contains a very important discussion of mathematics, logic, and the relationship between them.

3. See especially §434 of Russell [102], entitled "Mathematical reasoning requires no extra-logical element."

> the concept *in concreto*, although still not empirically, but only in an intuition which it presents a priori, that is, which it has constructed, and in which whatever follows from the general conditions of the construction must hold, in general, for the object [Objekte] of the concept thus constructed.
>
> Suppose a philosopher be given the concept of a triangle and he be left to find out, in his own way, what relation the sum of its angles bears to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, along with the concept of just as many angles. However long he meditates on these concepts, he will never produce anything new. He can analyse and clarify the concept of a straight line or of an angle or of the number three, but he can never arrive at any properties not already contained in these concepts. Now let the geometer take up this question. He at once begins by constructing a triangle. Since he knows that the sum of two right angles is exactly equal to the sum of all the adjacent angles which can be constructed from a single point on a straight line, he prolongs one side of the triangle and obtains two adjacent angles which together equal two right angles. He then divides the external angle by drawing a line parallel to the opposite side of the triangle, and observes that he has thus obtained an external adjacent angle which is equal to an internal angle—and so on. In this fashion, through a chain of inferences guided throughout by intuition, he arrives at a solution of the problem that is simultaneously fully evident [einleuchtend] and general. (A715–717/B743–745)

Kant is here outlining the standard Euclidean proof of the proposition that the sum of the angles of a triangle = 180° = two right angles ([46]: Book I, Prop. 32). Given a triangle *ABC*, one prolongs the side *BC* to *D* and then draws *CE* parallel to *AB* (see Figure 1). One then notes that $\alpha = \alpha'$ and $\beta = \beta'$, so $\alpha + \beta + \gamma = \alpha' + \beta' + \gamma = 180°$. Q.E.D.

In contending that construction in pure intuition is essential to this proof, Kant is making two claims that strike us as quite outlandish today. First, he is claiming that (an idealized version of) the figure we have drawn is necessary to the proof. The lines *AB*, *BD*, *CE*, and so on are indispensable constituents; without them the proof simply could not proceed. So geometrical proofs are themselves spatial objects. Second, it is equally important to Kant that the lines in question are actually drawn or continu-

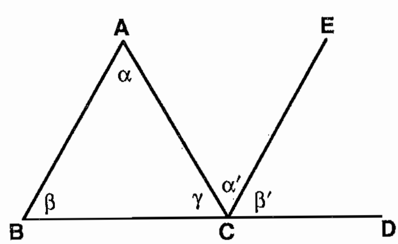

Figure 1

ously generated, as it were. Proofs are not only spatial objects, they are spatio-temporal objects as well. Thus, in an important passage in the Axioms of Intuition Kant says:

> I cannot represent to myself a line, however small, without drawing it in thought, that is gradually generating [nach und nach zu erzeugen] all its parts from a point. Only in this way can the intuition be obtained. . . . The mathematics of extension (geometry), together with its axioms, is based upon this successive synthesis of the productive imagination in the generation of figures [Gestalten]. (A162–163/B203–204)

That construction in pure intuition involves not only spatial objects, but also spatio-temporal objects (the motions of points), explains why intuition is able to supply a priori knowledge of (the pure part of) physics:

> . . . thus our idea of time [Zeitbegriff] explains the possibility of as much a priori cognition as is exhibited in the general doctrine of motion, and which is by no means unfruitful. (B49)

In other words, it is the spatio-temporal character of construction in pure intuition that enables Kant to give a philosophical foundation for both Euclidean geometry and Newtonian dynamics.

Kant's conception of geometrical proof is of course anathema to us. Spatial figures, however produced, are not essential constituents of proofs, but, at best, aids (and very possibly misleading ones) to the intuitive comprehension of proofs. Whatever the intended interpretation of the axioms or premises of a geometrical proof may be, the proof itself is a purely "formal" or "conceptual" object: ideally, a string of expressions in a given formal language. In particular, then, all that could possibly be missing from a purely "conceptual" or "analytic" derivation of ⌜*x*'s angles sum to 180°⌝ from ⌜*x* is a triangle⌝ are the *axioms* of Euclidean geometry. For us, the conjunction of ⌜*x* is a triangle⌝ with these axioms does of course imply ⌜*x*'s angles sum to 180°⌝ by logic alone; and no spatio-temporal activity of construction in pure intuition is necessary. To be sure, spatial objects may be needed to supply a particular interpretation of our axioms, but this is quite a different matter.

Is Kant simply forgetting about the axioms of Euclidean geometry here? This is most implausible, especially since the proof he sketches is Euclid's. No, his claim must be that even the conjunction of ⌜*x* is a triangle⌝ with these axioms does not imply ⌜*x*'s angles sum to 180°⌝ by logic alone: in other words, that Euclid's axioms do not imply Euclid's theorems by logic alone. Moreover, once we remember that Euclid's axioms are not the axioms used in modern formulations and, most important, that Kant's conception of logic is not our modern conception, it is easy to see that the claim in question is perfectly correct. For our logic, unlike Kant's, is

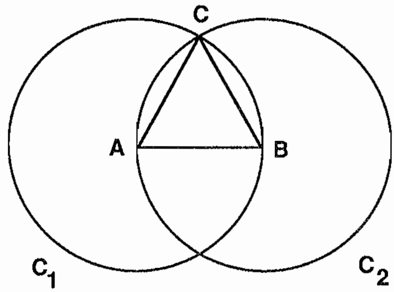

Figure 2

polyadic rather than monadic (syllogistic); and our axioms for Euclidean geometry[4] are strikingly different from Euclid's in containing an explicit, and essentially polyadic, *theory of order*.

The general point can be put as follows. A central difference between monadic logic and full polyadic logic is that the latter can generate an infinity of objects while the former cannot. More precisely, given any consistent set of monadic formulas involving k primitive predicates, we can find a model containing at most 2^k objects. In polyadic logic, on the other hand, we can easily construct formulas having only infinite models. Proof-theoretically, therefore, if we carry out deductions from a given theory using only monadic logic, we will be able to prove the existence of at most 2^k distinct objects: after a given finite point we will run out of "provably new" individual constants. Hence, monadic logic cannot serve as the basis for any serious mathematical theory, for any theory aiming to describe an infinity of objects (even "potentially").

This abstract and general point can be illustrated by Euclid's proof of the very first Proposition of Book I: that an equilateral triangle can be constructed with any given line segment as base. The proof runs as follows. Given line segment AB, construct (by Postulate 3) the circles C_1 and C_2 with AB as radius (see Figure 2). Let C be a point of intersection of C_1 and C_2, and draw lines AC and BC (by Postulate 1). Then, since (by the definition of a circle: Def. 15) $AC = AB = BC$, ABC is equilateral. Q.E.D.

There is a standard modern objection to this proof. Euclid has not proved the *existence* of point C; he has not shown that circles C_1 and C_2 actually intersect. Perhaps C_1 and C_2 somehow "slip through" one an-

4. These received their more-or-less definitive formulation in Hilbert [49]. I say "more-or-less" because there remains some confusion in Hilbert about the proper form of a continuity or completeness axiom.

other, and there is no point *C*. Moreover, in modern formulations of Euclidean geometry this "possibility" of non-intersection is explicitly excluded by a *continuity* axiom, an axiom which (apparently, anyway) does not appear in Euclid's list of Postulates and Common Notions. From this point of view, then, not only is Euclid's proof "defective," but so is his axiomatization: the existence of point *C* simply does not follow from Euclid's axioms.[5]

Why do we think that the existence of point *C* does not follow from Euclid's axioms? We might argue as follows. Cover the Euclidean plane with Cartesian coordinates in such a way that the midpoint of segment *AB* has coordinates (0, 0), point *A* has coordinates $(-1/2, 0)$, and point *B* has coordinates (1/2, 0). Then the desired point of intersection *C* has coordinates $(0, \sqrt{3}/2)$. Now throw away all points with irrational coordinates: the result is a model in Q^2, where Q is the rational numbers. This model appears to satisfy all Euclid's axioms, but, of course, point *C* does not exist in the model. So our model gives concrete form to the "possibility" of non-intersection, a "possibility" which therefore needs to be excluded by a continuity axiom.

But perhaps Euclid's formulation does contain such a continuity axiom, if only implicitly. After all, Postulate 2 states that straight line segments can be produced "continuously [κατὰ τὸ συνεχὲς]," while in our model straight lines are *dense* but not truly *continuous*. So one might think that our model is ruled out by Postulate 2. This attempt to "save" Euclid misses the central point. First, the intuitive notion of "continuity" figuring in Postulate 2 is not our notion of continuity: in particular, it is not explicitly distinguished from mere denseness. This distinction was not even articulated until late in the nineteenth century; before Dedekind mathematicians would commonly give what we call the definition of denseness when explaining what they meant by "continuity": namely, "for every element there is a smaller" or "between every two elements there is a third." Second, and more important, the notion of "continuity" in Postulate 2 is not logically analyzed: it appears as a simple (one-place) predicate. Therefore, whatever the intuitive meaning of "continuous" may be, there is certainly no valid *syllogistic inference* of the form:

$$\begin{array}{rl} & C_1 \text{ is continuous} \\ & C_2 \text{ is continuous} \\ \hline \therefore & C \text{ exists} \end{array}$$

5. A nice introductory account of such "defects" in Euclid is found in Eves [29], §8.1. Heath [46], vol. 1, pp. 234–240, provides a very detailed discussion of the "intersection" problem from a modern point of view. As far as I know the above criticism of Proposition I.1 was first made by Pasch [96], §6.

To get a valid inference of this form we need to analyze the notion of continuity in the modern style and to make an essential (and, as we shall see, rather strong) use of polyadic quantification theory.

Furthermore, once we start playing the game from a modern point of view we can generate trivial "counter-examples" to Euclid that do not depend on sophisticated considerations like continuity. Thus, in Figure 2, throw away all points of the plane except the two points *A*, *B*. Let the "line" *AB* be just the pair {*A*, *B*}, let the "circle" C_1 be the singleton {*B*}, and let the "circle" C_2 be the singleton {*A*}. Does "line" *AB* satisfy Postulate 2? Can it be "produced continuously"? Note again that neither "can be produced" nor "continuous" is logically analyzed: both appear as simple (one-place) predicates. So, from a strictly logical point of view, we can give them both any interpretation we like: let them both mean "has two elements," for example. Then Postulate 2 is obviously satisfied, and so are the other axioms. Hence, Euclid's axiomatization does not even imply the existence of more than two points.

Does this last "counter-example" show that Euclid's axiomatization is hopelessly "defective"? I think not. Rather, it underscores the fact that Euclid's system is not an axiomatic theory in our sense at all. Specifically, the existence of the necessary points is not logically deduced from appropriate existential axioms. Since the set of such points is of course infinite, this procedure cannot possibly work in a monadic (syllogistic) context. Instead, Euclid *generates* the necessary points by a definite process of construction: the procedure of construction with straight-edge and compass. We start with three basic operations: (i) drawing a line segment connecting any two given points (to avoid complete triviality we assume two distinct points to begin with), (ii) extending a line segment by any given line segment, (iii) drawing a circle with any given point as center and any given line segment as radius. We are then allowed to iterate operations (i), (ii), and (iii) any finite number of times. Euclid's Postulates 1–3 give the rules for this iterative procedure, and the points in our "model" are just the points that can be so constructed. In particular, then, the infinity of this set of points is guaranteed by the infinite iterability of our process of construction.[6]

More precisely, it is straightforward to show that the points generated by straight-edge and compass constructions (and, therefore, the points required for Euclidean geometry) comprise a Cartesian space (set of pairs) based on the so-called square-root (or "Euclidean") extension Q* of the

6. See Eves [29], chap. IV, for a discussion of the mathematics of Euclidean constructions. Compare also the very helpful contrast between the Euclidean approach to existence and the modern approach typified by Hilbert in Mueller [81], pp. 11–15. I am indebted to William Tait for emphasizing the importance of straight-edge and compass constructions to me, and for helping me to get clearer about their essential properties.

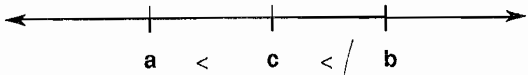

Figure 3

rationals, where Q* results from closing the rationals under the operation of taking real square-roots (see Eves [29], §9.3). In particular, then, the underlying set (Q* × Q*) is only a small fragment of the full Cartesian plane $\mathbf{R}^2$, where **R** is the real numbers. The former, unlike the latter, is a denumerable set, and each element is determined by a finite sequence of elementary operations. In this sense, there is no need in Euclidean geometry for anything as strong as a continuity axiom.

Compare Euclid's approach to the existence of points—in particular, to the existence of an infinity of points—with that taken by modern axiomatizations. The basis of the modern approach, beginning with Pasch in 1882 and culminating in Hilbert's *Foundations of Geometry* (1899), is to include an explicit *theory of order:* a theory of the order structure (and cardinality) of the points on a line. Thus, imagine the points on any line to be ordered by a two-place relation < of "being-to-the-left-of" (see Figure 3). Governing < is the theory of dense linear order without endpoints:

1. $\neg(a < a)$	(irreflexivity)
2. $a < c \,\&\, c < b \rightarrow a < b$	(transitivity)
3. $a < b \vee b < a \vee a = b$	(connectedness)
4. $\forall a \exists b\, (a < b)$	
5. $\forall b \exists a\, (a < b)$	(no endpoints)
6. $\forall a \forall b \exists c\, (a < b \rightarrow (a < c < b))$	(denseness)

The presence of some such axioms as 1–6 is the chief difference between Hilbert's axiomatization and Euclid's.[7]

7. I have left out the continuity axiom (which is of course second-order), so axioms 1–6 will be satisfied in the rationals Q. The resulting Cartesian space Q^2 will therefore be insufficient for Euclidean geometry. Nevertheless, as noted above, full continuity is certainly not required, and it suffices to supplement the Cartesian space based on axioms 1–6 with an axiom of intersection for straight lines and circles (this, of course, is where the square-roots come in). See the excellent survey by A. Tarski, "What is Elementary Geometry?" (1959)—reprinted in Hintikka [51]. In particular, Tarski gives a set of axioms sufficient to generate a Cartesian space based on the square-root extension Q* (see *Theorem 6* governing system $\mathscr{E}_2''$: the circle/line axiom is A13′ on p. 174 of [51]; like the denseness condition it has the logical form $\forall .\, .\forall\exists$). Adding a (first-order) continuity *schema* extends our underlying set to a real closed field (see *Theorem 1* governing system $\mathscr{E}_2$: the continuity schema is A13 on p. 167 of [51]; it has the (minimal) logical form $\forall .\, .\forall\exists\exists\exists\forall\forall$). Finally, adding a (second-order) continuity *axiom* gives us a system essentially equivalent to Hilbert's: the underlying set is precisely $\mathbf{R}^2$.

Axioms 1–6 have only infinite models, and, of course, they make an essential use of modern polyadic logic. Note, however, that it is not merely the presence of two-place as opposed to one-place predicates that is crucial here. After all, axioms 1–3 alone certainly have finite models. Rather, the essential new element is the *quantifier-dependence* exhibited in 4–6: the logical form $\forall x\ \exists y$.[8] This kind of dependence of one quantifier on another cannot arise in monadic logic, where we can always "drive quantifiers in" so that each one-place matrix is governed by a single quantifier. (Thus, for example, $\forall x\ \exists y\ (Fx \rightarrow Gy)$ is equivalent to $\forall x\ Fx \rightarrow \exists y\ Gy$.) Moreover, it is the dependence of one quantifier on another—specifically, of existential quantifiers on universal quantifiers—that enables us to capture the intuitive idea of an iterative process formally: any value x of the universal quantifier generates a value y of the existential quantifier, y can then be substituted for x generating a new value y', and so on. Hence, the existence of an infinity of objects can be deduced explicitly by logic alone.

We can now begin to see what Kant is getting at in his doctrine of construction in pure intuition. For Kant logic is of course syllogistic logic or (a fragment of) what we call monadic logic.[9] Hence for Kant, one cannot represent or capture the idea of infinity formally or conceptually: one cannot represent the infinity of points on a line by a formal theory such as 1–6 above. If logic is monadic, one can only represent such infinity intuitively—by an iterative process of spatial construction:

> Space is represented as an infinite given magnitude. A general concept of space (which is common to both a foot and an ell alike) can determine nothing in regard to magnitude [Größe]. Were there no limitlessness in the

8. Thus, Euclid's Common Notions contain axioms governing an *equality* or *congruence* relation and axioms governing the *part-whole* relation. The point, however, is that such axioms are "essentially monadic" in exhibiting no quantifier-dependence (we could formulate them using universal free-variables as in axioms 1–3). Moreover, these Euclidean axioms have finite models: they do not say anything about the cardinality of our underlying set of points. Interestingly enough, Kant explicitly says that these axioms are analytic: cf. B16–17, A164/B203. (For more on the notion of "essentially monadic" and Kant's conception of analyticity see note 14 below.)

9. Kant's actual views on logic involve many subtleties which I here pass over. See, in particular, the interesting discussion in Thompson [108]. Thompson argues very convincingly that Kant indeed made one substantial advance in logic by replacing the traditional logic of *terms* with a "transcendental logic" of *objects* and *concepts:* "a logic in which the form of predication is '*Fx*' and not '*S* is *P*' " ([108], p. 342). Yet I cannot follow Thompson when he says: "The general logic required by Kant's transcendental logic is thus at least first-order quantificational logic plus identity" ([108], p. 334). If we do not limit ourselves to the logical forms of traditional syllogistic logic, Kant's Table of Judgements makes no sense. It is more plausible, I think, to equate Kant's conception of logic with, at most, *monadic* (or perhaps "essentially monadic"—see note 8 above) quantification theory plus identity (which, as far as I can see, is all Thompson requires in his fascinating discussion of Kant, Strawson, and Quine on singular terms and descriptions: [108], pp. 334–335—especially n. 15).

progression of intuition, no concept of relations could, by itself, supply a principle of their infinitude. (A25)

. . . that one can require a line to be drawn to infinity *(in indefinitum)*, or that a series of changes (for example, spaces traversed by motion) shall be infinitely continued, presupposes a representation of space and time that can only depend on intuition, namely, in so far as it in itself is bounded by nothing; for from concepts alone it could never be inferred. (*Prolegomena* §12: 4, 285.1–7)

Space is represented as an infinite given quantity [Größe]. Now one must certainly think every concept as a representation which is contained in an infinite aggregate [Menge] of different possible representations (as their common characteristic [Merkmal]), and it therefore contains these *under itself*. But no concept, as such, can be so thought as if it were to contain an infinite aggregate of representations *in itself*. Space is thought in precisely this way, however (for all parts of space *in infinitum* exist simultaneously). Therefore the original representation of space is an a priori *intuition*, and not a *concept*. (B40)

Kant's point is that (monadic) conceptual representation is quite inadequate for the representation of infinity: (monadic) concepts can never contain an infinity of objects in their very idea, as it were. In particular, then, since our idea of space does have this latter property, it cannot be a (monadic) concept.[10]

The notion of infinite divisibility or denseness, for example, cannot be represented by any such formula as 6: this logical form simply does not exist. Rather, denseness is represented by a definite fact about my intuitive capacities: namely, whenever I can represent (construct) two distinct points a and b on a line, I can represent (construct) a third point c between them. Pure intuition—specifically, the iterability of intuitive constructions[11]—provides a uniform method for instantiating the existential quantifiers we would use in formulas like 6; it therefore allows us to capture notions like denseness without actually using quantifier-dependence. Before the invention of polyadic quantification theory there simply is no alternative.

Thus, in Euclid's geometry there is no axiom corresponding to our denseness condition 6. Instead, we are given a uniform method for actually

10. I am indebted to Manley Thompson for correcting my earlier discussion of B40 in which I uncritically assimilated Kant's notion of the extension of a concept to our own (as well as for correcting my earlier mistranslation of the second sentence of B40).

11. For the centrality of *indefinite iterability* to Kant's conception of pure intuition, I am indebted, above all, to Parsons, "Kant's Philosophy of Arithmetic," especially §VII. But see also Parsons, "Infinity and Kant's Conception of the 'Possibility of Experience' " for doubts about the "psychological" or "empirical" reality of such truly *indefinite* iterability. Space prevents me from here giving these doubts the extended discussion they deserve.

constructing the point bisecting any given finite line segment: it suffices to join C in the Proof of Proposition I.1 with its "mirror image" below AB—the resulting straight line bisects AB (Prop. I.10). This operation, which is itself constructed by iterating the basic operations (i), (ii), and (iii), can then be iterated as many times as we wish, and infinite divisibility is thereby represented. So we do not derive new points between A and B from an existential axiom, we construct a bisection function from our basic operations and obtain the new points as the values of this function:[12] in short, we are given what modern logic calls a *Skolem function* for the existential quantifier in 6.[13] For Kant, this procedure of generating new points by the iterative application of constructive functions takes the place, as it were, of our use of intricate rules of quantification theory such as existential instantiation. Since the methods involved go far beyond the essentially monadic logic available to Kant, he views the inferences in question as synthetic rather than analytic.[14]

12. A simpler illustration of these ideas is provided by the theory of successor based on a constant 0 and a one-place function-sign $s(x)$. Instead of saying "Every number has a successor," we lay down the axioms:

$$0 \neq s(x)$$

$$s(x) = s(y) \rightarrow x = y.$$

These axioms have only infinite models, for we have "hidden" the quantifier-dependence in the function-sign $s(x)$: we *presuppose* that the corresponding function is well defined for all arguments.

13. A Skolem function for y in $\forall x\, \exists y\, R(x, y)$ is a function $f(x)$ such that $\forall x\, R(x, f(x))$; Skolem functions for y, w in $\forall x\, \exists y\, \forall z\, \exists w\, B(x, y, z, w)$ are functions $f(x)$, $g(x, z)$ such that $\forall x\, \forall z\, B(x, f(x), z, g(x, z))$; and so on. See, for example, Enderton [22], §4.2. (Here I follow a suggestion by Thomas Ricketts.)

14. These ideas have much in common with Hintikka's reconstruction in [52]. As in the present account, Hintikka argues that Kant's analytic/synthetic distinction is drawn *within* what we now call quantification theory, and Hintikka calls a quantificational argument *synthetic* when (roughly) "new individuals are introduced." Thus, synthetic arguments, for Hintikka, will correspond closely to those in which Skolem functions figure essentially, and analytic arguments will correspond to those we are calling "essentially monadic." Hintikka also notes the importance, in this connection, of the (often ignored) fact that Kant's logic is syllogistic or monadic ([52], pp. 189–190). (In this respect, Hintikka has indeed made an important advance over Beth. For Beth considers only the trivial procedure of conditional proof followed by universal generalization, and therefore puts forward a conception of the role of "intuition" in proof that applies equally well to monadic or syllogistic logic: cf. [6], §§5–7.) Yet Hintikka views the problem of quantificational rules like existential instantiation in rather the wrong light, I think—particularly when he attempts to conceive Kant's "transcendental method" as, in part, a *justification* of such rules (see chap. V of [52]). As I understand it, the whole point of pure intuition is to enable us *to avoid* rules of existential instantiation by actually constructing the desired instances: we do not derive our "new individuals" from existential premises but construct them from previously given individuals via Skolem functions.

Finally, we should note that our modern distinction between pure and applied geometry, between an uninterpreted formal system and an interpretation that makes such a system true, cannot be drawn here. In particular, the only way to represent the theory of linear order 1–6 is to provide, in effect, an interpretation that makes it true.[15] The idea of infinite divisibility or denseness is not capturable by a formula or sentence, but only by an intuitive procedure that is itself dense in the appropriate respect. By the same token, the sense in which geometry is a priori for Kant is also clarified. Thus, the proposition that space is infinitely divisible is a priori because its truth—the existence of an appropriate "model"—is a condition for its very possibility.[16] One simply cannot separate the idea or representation of infinite divisibility from what we would now call a model or realization of that idea; and our notion of pure (or formal) geometry would have no meaning whatsoever for Kant. (In a monadic context a pure or uninterpreted "geometry" cannot be a geometry at all, for it cannot represent even the *idea* of an infinity of points.)

· II ·

The above considerations make a certain amount of sense out of Kant's theory, but one might very well have doubts about attributing them to Kant. After all, Kant certainly had no knowledge of the distinction between monadic and polyadic logic, nor of quantifier-dependence, Skolem functions, and so on. So using such ideas to explicate his theory may appear wildly anachronistic, and my reading of the passages from A25, *Prolegomena* §12, and B40 may appear strained. Thus, whereas in all these passages Kant does clearly state that general concepts are inadequate for the representation of infinity and does contrast purely conceptual representation with the unlimited or indefinite iterability of pure intuition, it is not at all clear that this inadequacy, for Kant, rests on the limitations of monadic or syllogistic logic. Indeed, it is very hard to see how Kant could possibly have comprehended what we would now express as the inadequacy of monadic logic.

15. Similarly, the theory of successor of note 12 contains an infinite sequence of terms—the so-called numerals 0, s(0), s(s((0)), and so on—that is itself a model for that theory. Compare Parsons, "Kant's Philosophy of Arithmetic," §VII, and Thompson [108], pp. 337–342, where this feature of the numerals is connected with Kant's views on "symbolic construction."

16. I am indebted to Philip Kitcher for prompting me to make this last point explicit. The proposition that *physical* space is infinitely divisible is quite a different matter, however, whose a priori truth requires transcendental deduction: see note 32 and §IV below. For further discussion of the precise sense in which geometry is a priori for Kant see Chapter 2.

The key passage in this connection is B40, for it is only here that Kant does more than simply assert the inadequacy of general concepts (and hence the need for pure intuition) in the representation of infinity: it is only here that Kant attempts to explain precisely what it is about general concepts that is responsible for this inadequacy. Now B40 operates with Kant's particular notions of the extension and intension of a concept.[17] The extension of a concept is the totality of concepts relating to it as species, subspecies, and so on to a higher genus: the extension of *body* includes *animate body, inanimate body, animal animate body, rational animal animate body,* and so forth. In Kant's terminology these species, subspecies, and so on are all contained *under* the given concept. For Kant, moreover, there is no lowest *(infima)* species: our search for narrower and narrower specifications of any give (empirical) concept necessarily proceeds without end (A654–656/B682–684). In this sense, the extension of a concept is always unlimited. The intension of a concept, on the other hand, is the totality of constituent concepts (Teilbegriffe) or characteristics (Merkmale) that occur in its definition: the intension of *man* thus includes *rational, animal, animate,* and *body*. In Kant's terminology these constituent concepts or characteristics are all contained *in* the given concept.

Kant is therefore making two basic points in B40. First, the extension of a concept is always unlimited or potentially infinite: "one must certainly think every concept as a representation which is contained in an infinite aggregate of different possible representations (as their common characteristic), and it therefore contains these *under itself*." Nevertheless, however, no given concept can be conceived as the conjunction of an infinite number of constituent concepts. There is no bound to the number of elements in a concept's intension, but this number is always finite. In brief: whereas extensions are always unlimited, intensions can never be infinite—as, for example, the intension of a Leibnizean complete concept would be (compare Allison [2], p. 93).

Yet this way of expressing the matter makes it appear that the point of B40 has nothing whatever to do with the inadequacy of monadic logic for representing the idea of infinity. For the latter depends on the fact that no set of monadic formulas has only infinite models: any satisfiable set of monadic formulas is satisfiable in a domain consisting of only a finite number of objects. What is at issue here, then, is a fact about extensions of concepts *in the modern sense:* no concept definable by purely monadic

17. See the *Jäsche Logik,* Part I, Section One: 9, 91–100. See also the illuminating discussion of B40 in Allison [2], pp. 92–94. Again, I am especially indebted to Manley Thompson for emphasizing the importance of Kant's particular notions of extension and intension to me.

concepts species relation

contra relation

means can force its extension—that is, the set of objects falling under it—to be infinite. And not only is the modern notion of the extension of a concept completely foreign to Kant (Kant's notion involves a relation between a concept and other concepts—its species, subspecies, and so on—rather than a relation between a concept and the objects falling under it), but Kant explicitly states that extensions in his sense are potentially infinite. To be sure, he also explicitly states that intensions are necessarily finite; but what does this have to do with our modern fact about finite extensions?

It seems to me that there is nonetheless an intimate connection indeed between our modern fact about the limitations of monadic concepts and what Kant is saying in B40. For how do we establish that, for example, no set of monadic formulas containing k primitive predicates can determine a model with more than 2^k objects? We observe that k primitive predicates $P_1, P_2, \ldots, P_k$ can partition the domain into only 2^k maximally specific subclasses: the classes of objects that are P_1 & P_2 & . . . & P_k, $\neg P_1$ & P_2 & . . . & P_k, P_1 & $\neg P_2$ & . . . & P_k, and so on. The number of distinct objects we can assert to exist, then, is bounded by 2^k: we can say that there is an object in the first partition, there is an object in the second partition, and so on—and this is all. But now each such partition corresponds to a Kantian intension of a concept: the concept defined by the given conjunction of primitive predicates (and their negations). Moreover, each primitive predicate includes all such conjunctions in which it occurs in its Kantian extension (together with all less specific conjunctions in which it occurs, of course). What Kant is saying in B40 is that, whereas there is no limit to the number of (empirical) concepts we may eventually introduce, we are operating at any given time with only a finite number: we are therefore never in a position to form an infinite conjunction (an infinite intension). Similarly, on a modern understanding of monadic logic, since the number of primitive predicates is always finite (although it can be as large as one pleases), the number of partitions of the domain one can construct by conjoining such predicates (and their negations) is also always finite: we are therefore never in a position to assert the existence of an infinite number of objects. It seems to me, then, that the situation can be fairly described as follows: although there can of course be no question of Kant explicitly comprehending the logical fact we would now express as the inadequacy of monadic logic for representing an infinity of objects, he nonetheless comes as close to this as is possible given his own understanding of logic.

This reading of B40 also illuminates its position and role within the general argument of the Metaphysical Exposition of the Concept of Space. The passage at B40 is of course the concluding paragraph of the argument for the intuitive character of our representation of space—an argument which begins at A24–25/B39:

> Space is not a discursive or, as one says, general concept of relations of things as such, but a pure intuition. For, first, one can represent to oneself only a single space; and if one speaks of several spaces, one means thereby only parts of one and the same unique space.

Kant begins, then, by asserting that space is a singular individual rather than a general concept: the various particular spaces do not relate to space as instances to a general concept but as parts to an individual whole.

However, as Kant explicitly asserts at A25, there is indeed a "general concept of space (which is common to both a foot and an ell alike)," and this concept of space—which we might represent by ⌜*x* is a space⌝—does relate to parts of space or spaces as general concept to the instances thereof. In other words, there is both the general concept ⌜*x* is a space⌝, of which particular spaces are instances, and the singular individual *space,* of which particular spaces are parts (and which is in turn itself an instance of the general concept ⌜*x* is a space⌝). The question is: why should the latter have priority over the former? Why should our idea or representation of space be identified with the singular individual *space* rather than the general concept ⌜*x* is a space⌝?

Kant's answer is given in the following three sentences:

> Nor can these parts precede the single all-inclusive space, as being, as it were, its constituents (and making its composition possible); on the contrary, they can be thought only *in it.* Space is essentially singular: the manifold in it, and hence the general concept of spaces as such, rests purely on limitations. It follows therefrom that an a priori intuition (that is not empirical) underlies all concepts of space. (A25/B39)

One cannot arrive at the singular individual *space* by starting from the general concept ⌜*x* is a space⌝—which includes all parts of space among its instances—and, as it were, assembling the individual *space* from these diverse spaces or parts of space. On the contrary, the only way to arrive at the general concept ⌜*x* is a space⌝ is via the intuitive act of "cutting out" parts of space from the singular individual *space.* It is only the latter intuitive procedure of "limitation" that makes the general concept of space and of spaces possible in the first place.

But now the question becomes why should this be so: Why should the singular intuition thus precede the general concept? In the sentence immediately following Kant appeals to our knowledge of geometry:

> So too are all principles [Grundsätze] of geometry—for example, that in a triangle two sides together are greater than the third—derived: never from

> general concepts of line and triangle, but only from intuition, and this indeed a priori, with apodictic certainty.

In the end, therefore, Kant's claim of priority for the singular intuition *space* rests on our knowledge of geometry.[18] Our cognitive grasp of the notion of space is manifested, above all, in our geometrical knowledge. Hence, if we can show that this knowledge is intuitive rather than conceptual, we will have shown the inadequacy of the general concept of space and the priority of the singular intuition.

Continuing our line of questions, then, we must ask why, at bottom, is conceptual knowledge inadequate to geometry: why must intuition play an essential role? Surely the mere assertion that geometrical principles cannot be derived "from general concepts . . . but only from intuition" is not expected to convince those who, like the Leibnizeans and Wolffians, maintain precisely the opposite. It is at this crucial point that Kant inserts the argument of B40. What is required for establishing the intuitive character of our representation of space is not simply the fact that space consists of parts, but rather—as geometry demonstrates—the fact that it consists of an *infinite number* of parts: "all parts of space *in infinitum* exist simultaneously." Thus, for example, geometry shows us that space is divisible into a potentially infinite sequence of smaller and smaller parts;[19] and, as the argument of B40 makes clear, no mere (monadic) concept can possibly capture this essential feature of our representation of space.

Once again, Kant's conception of infinity and infinite divisibility can be clarified by contrasting it with modern formulations. We, of course, can easily represent infinite divisibility by means of *(polyadic)* concepts—as we did above in the theory of dense linear order. In such a theory the points on a line are taken as primitive, and the line itself is built up from them in just the way Kant says it cannot be: the points relate to the line as "its constituents (and making its composition possible)." Yet what makes this representation itself possible is precisely the quantifier-dependence of modern polyadic logic: the logical form $\forall \ldots \forall\exists$. In the

18. But see also the very interesting account in Melnick [77], chap. 1.A, emphasizing the *individuating* role of the singular intuition. Thus, for example, two cubic feet of space are not distinguished by the general concept ⌜*x* is a space⌝ (or by any other general concept), but only by their "positions" in *space* ([77], pp. 9–14). In this connection, see also Thompson, [108]. Nevertheless, although Kant does emphasize this individuating role of the singular intuition in various places (particularly at A263–264/B319–320 and A271–272/B327–328), there is no hint of such a role at A24–25/B39. On the contrary, the emphasis *here* is entirely on the priority of the intuitive procedure of "limitation."

19. This fact of geometry plays a central role for Kant in his opposition to Leibnizean-Wolffian metaphysics: see, in particular, the *Physical Monadology,* especially Proposition III (1, 478–479). Compare also A165–166/B206–207. See the Introduction above.

absence of such logical forms—and in accordance with the actual procedure of Euclid's geometry—the natural alternative is to represent infinite divisibility by an intuitive constructive procedure for "cutting out" a smaller line segment from any given one: for example, Euclid's construction for bisecting a line segment of Proposition I.10.

Thus, whereas we can represent infinite divisibility by $\ulcorner\forall x\ \exists y\ (y \text{ is a proper part of } x)\urcorner$, Kant would formulate this proposition by $\ulcorner f_B(x)$ is a proper part of $x\urcorner$, where $f_B(x)$ is the operation of bisection, say.[20] And, in this representation, the idea of infinity is conveyed not by logical features of the relational concept $\ulcorner y$ is a proper part of $x\urcorner$ but by the well-definedness and iterability of the function $f_B(x)$: our ability, for any given line segment x, to construct (distinct) $f_B(x)$, $f_B(f_B(x))$, *ad infinitum*.[21] This, I suggest, is why Kant gives priority to the singular intuition *space*, from which all parts or spaces must be "cut out" by intuitive construction ("limitation"). Only the unbounded iterability of such constructive procedures makes the idea of infinity, and therefore all "general concepts of space," possible. And, of course, it is this very same constructive iterability that underlies the proof-procedure of Euclid's geometry.

· III ·

Even if we are on the right track, however, we have still gone only part of the way towards understanding construction in pure intuition. We can bring out what is missing by three related observations. First, as we noted above, the notions of denseness, infinite divisibility, and (even) constructibility with straight-edge and compass do not amount to full continuity.

20. Kant mentions the Euclidean construction of the bisection operation in the *Groundwork of the Metaphysics of Morals* at 4, 417.18–21; in the *Physical Monadology,* however, he uses a different proof of infinite divisibility due to John Keill. (We understand the output of $f_B(x)$ here to be not the midpoint of x but, say, the left-most half segment of x.)

21. Kant certainly recognizes relations such as the part-whole relation. For Kant, however, our theory of this relation will still be "essentially monadic" in exhibiting no quantifier-dependence: see notes 8 and 14 above. In particular, then, the inference from $\ulcorner f_B(x)$ is a proper part of $x\urcorner$ to $\ulcorner f_B(f_B(x))$ is a proper part of $f_B(x)\urcorner$ is synthetic for Kant (= not "essentially monadic") because (in Hintikka's terminology) "new individuals are introduced." Moreover, it is instructive to contrast this inference with the following example from Leibniz's *New Essays:* If Jesus Christ is God, then the mother of Jesus Christ is the mother of God ([67], vol. 5, p. 461; [71], p. 479—compare the citation in Tait [107], §XIII). If we represent motherhood as a function (and hence as presupposing existence) then Leibniz's argument also "introduces new individuals" and thus should count as synthetic; if we represent motherhood as a relation (and hence as not presupposing existence) then the argument counts as "essentially monadic" or analytic. Since there is no question of *iteration* in Leibniz's argument, however, the functional representation is certainly not required, and it is just this, it seems to me, that distinguishes the argument from $\ulcorner f_B(x)$ is a proper part of $x\urcorner$ to the infinite divisibility of space.

These notions all involve denumerable sets of points which are but small fragments of the set **R** of real numbers. Hence, to understand how full continuity comes in we have to go beyond Euclidean geometry. Second, these notions do not (on a modern construal) exploit very much of polyadic logic: just the logical form $\forall \ldots \forall\exists$; if this were all that was required modern logic would hardly need to have been invented. Third, the procedure of construction with Euclidean tools—with straight-edge and compass—does not really exploit the kinematic element that is essential to Kant's conception of pure intuition: no appeal is made to the idea that lines, circles, and so on are generated by the *motion* of points. So why does Kant think that *motion* is so important?

These three observations are in fact intimately related. For there exists a branch of mathematics which was just being developed in the seventeenth and eighteenth centuries; which does require genuine continuity—"all" or "most" real numbers; whose modern, "rigorous" formulation requires full polyadic logic—much more intricate forms of quantifier-dependence than $\forall \ldots \forall\exists$; and, finally, whose earlier, "non-rigorous" formulation made an essential appeal (in at least one tradition) to temporal or kinematic ideas—to the intuitive idea of motion. This branch of mathematics is of course the calculus, or what we now call real analysis. It goes far beyond Euclidean geometry in considering "arbitrary" curves or figures—not merely those constructible with Euclidean tools—and in making extensive use of *limit operations.*

From a modern point of view the basic limit operation underlying the calculus is explained in terms of the Cauchy-Bolzano-Weierstrass notion of *convergence*. Moreover, we also appeal to this notion in explaining the distinction between denseness and genuine continuity, in precisely expressing the idea that there are "gaps" in a merely dense set such as the rational numbers. Thus, let $s_1, s_2, \ldots$ be a sequence of rational numbers that converges to π, say—that approaches π as its limit (for example, let $s_1 = 3.1$, $s_2 = 3.14$, and in general s_n = the decimal expansion of π carried out to n places). This sequence of rationals *converges* (to "something," as it were), but in the set Q of rational numbers (and even in the expanded set Q* of Euclidean-constructible numbers) there is no limit point it *converges to.* Such limit points are "missing" from a merely dense set such as the rationals. A truly continuous set contains "all" such limit points.

More precisely, using Cauchy's criterion of 1829, we say that a sequence $s_1, s_2, \ldots$ *converges* if

$$\forall\epsilon\ \exists N\ \forall m\ \forall n\ [m,n > N \rightarrow |s_m - s_n| < \epsilon],$$

where ϵ is a positive rational number and N, m, n are natural numbers.

A sequence $s_1, s_2, \ldots$ *converges to a limit r* if

$$\forall \epsilon\ \exists N\ \forall m\ [m > N \rightarrow |s_m - r| < \epsilon].$$

The problem with a merely dense order is that the first can be true even when the second is not, whereas a continuous order satisfies the additional axiom of *Cauchy completeness*—whenever a sequence converges, it converges to a limit r—which clearly has the logical form $\forall\exists\forall\forall \rightarrow \exists\forall\exists\forall$. Note the additional logical complexity of this axiom: in particular, the use of the strong form of quantifier-dependence, $\forall\exists\forall$.

The increase in logical strength which I find so striking here can be best brought out if we compare the way points are generated by a completeness axiom with the way they are generated by Euclidean constructions (or, from a modern point of view, by the weaker form of quantifier-dependence, $\forall.\ .\forall\exists$). In the latter case, although the total number of points generated is of course infinite, each particular point is generated by a finite number of iterations: each point is determined by a finite number of previously constructed points. In generating or constructing points by a limit operation, on the other hand, we require an infinite sequence of previously given points: no finite number of iterations will suffice. So limit operations involve a much stronger and more problematic use of the notion of infinity than that involved in a simple process of iterated construction.

Let us now return to Kant and the late eighteenth century. We cannot of course represent the ideas of convergence and transition to the limit by complex quantificational forms such as $\forall\exists\forall$. But the idea of *continuous motion* appears to present us with a natural alternative. Thus, for example, we can easily "construct" a line of length π by imagining a continuous process that takes one unit of time and is such that at $t = 1/2$ a line of length 3.1 is constructed, at $t = 2/3$ a line of length 3.14 is constructed, and in general at $t = n/(n + 1)$ a line of length s_n is constructed, where s_n, again, equals the decimal expansion of π carried out to n places. Assuming this process in fact *has* a terminal outcome, at $t = 1$ we have constructed a line of length π. In this sense, then, we can thereby "construct" any real number.[22]

Here the notion of convergence or approach to the limit is expressed by a temporal process: by the idea of one point moving or becoming closer and closer to a second. This intuitive process of becoming does the work of our logical form $\forall\exists\forall$, as it were. That the limit of a convergent se-

22. Here, and throughout this section especially, I am indebted to clarificatory suggestions from Roberto Torretti.

quence exists is expressed by the idea that any such process of temporal generation has a terminal outcome. This idea does the work of our logical form $\exists\forall\exists\forall$. In particular, then, what we now call the continuity or completeness of the points on a line is expressed by the idea that any finite motion of a point beginning at a definite point on our line also stops at a definite point on our line.[23] What the modern definition of convergence does, in effect, is replace this intuitive conception based on motion and becoming with a formal, algebraic, or "static" counterpart based on quantifier-dependence and order relations.

Now a temporal conception of the limit operation is explicit in the basic lemma Newton uses to justify the mathematical reasoning of *Principia:*

> Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal. (Book I, §1, Lemma I: [82], p. 73; [83], p. 29)

This lemma is perhaps best understood as a definition of what Newton means by "quantity": namely, an entity generated by a continuous temporal process (it clearly fails for discontinuous "quantities") .

Newton's conception of "quantities" as temporally generated is even more explicit, of course, in his method of *fluxions,* where all mathematical entities are thought of as *fluents* or "flowing quantities." For example:

> I don't here consider Mathematical Quantities as composed of Parts *extremely small,* but as *generated by a continual motion.* Lines are described, and by describing are generated, not by any apposition of Parts, but by a continual motion of Points. Surfaces are generated by the motion of Lines, Solids by the motion of Surfaces, Angles by the Rotation of their Legs, Time by a continual flux, and so in the rest. ([85], p. 141)

Moreover, Kant appears to be echoing these ideas in an important passage about continuity in the Anticipations of Perception:

> Space and time are *quanta continua,* because no part of them can be given without being enclosed between limits (points and instants), and therefore only in such fashion that this part is itself again a space or a time. Space consists only of spaces, time consists only of times. Points and instants are only limits, that is, mere places [Stellen] of their limitation. But places always presuppose the intuitions which they limit or determine; and out of mere places, viewed as constituents capable of being given prior to space or time, neither space nor time can be composed [zusammengesetzt]. Such quantities

23. Compare Heath's interpretation of the intuitive content of Dedekind continuity: "[it] may be said to correspond to the intuitive notion which we have that, if in a segment of a straight line two points start from the ends and describe the segment in opposite senses, they meet in a point" ([46], p. 236).

> [Größen] may also be called *flowing* [*fließende*], since the synthesis (of the productive imagination) in their generation [Erzeugung] is a progression in time, whose continuity is most properly designated by the expression of flowing (flowing away). (A169–170/B211–212)

For Kant, like Newton, spatial quantities are not composed of points, but rather generated by the motion of points.

I take Kant's choice of language to be especially significant here, for his "fließende Größen" is the standard German equivalent of Newton's "fluents."[24] This expression is used, for example, by the mathematician Abraham Kästner in his influential textbooks on analysis and mathematical physics.[25] Kästner's analysis text attempts to develop the calculus from a "rigorous" standpoint that makes no appeal to infinitely small quantities. In this connection he develops a version of Newton's method of fluxions, and, what is more remarkable for a German author of this period, he argues that Newton's fluxions are in some respects clearer and more perspicuous than Leibniz's differentials. Further, he explicitly applauds Collin Maclaurin's attempt, in his monumental *Treatise of Fluxions* (1742), to develop the calculus on the basis of a kinematic conception of the limit operation.[26]

Without going into detail, the most basic ideas of the fluxional calculus are as follows.[27] We start with fluents or "flowing quantities" x, y, conceived as continuous functions of time. We can then form the fluxions or time-derivatives $\dot{x}$, $\dot{y}$, because continuously changing quantities obviously have well-defined instantaneous velocities or rates of change.[28] If we are then given a curve or figure $y = f(x)$ generated by independent motions in rectangular coordinates of the fluents x, y, the *derivative* (slope of the tangent line) will be $dy/dx = \dot{y}/\dot{x}$ (by the "parallelogram of velocities") (see Figure 4). Finally, we can recover the integral from the derivative via the Fundamental Theorem (which is also understood temporally: see

24. Kitcher, in [59], p. 41, has drawn attention to Kant's use of Newtonian terminology and the connection between B211–212 and the fluxional calculus. See also note 29 below.

25. See *Anfangsgründe der Analysis des Unendlichen* (Göttingen, 1761) and *Anfangsgründe der höhern Mechanik* (Göttingen, 1766). Kant was well acquainted with Kästner's works, and an admirer of them. See the reference to Kästner's 1766 work in §14.4 of the *Inaugural Dissertation*, for example: 2, 400.3–4.

26. Philip Kitcher has emphasized to me that Maclaurin's book contains a number of different approaches to the foundations of the calculus. In Maclaurin—and in the Newtonian tradition generally—it is certainly not obvious that the kinematic approach is predominant. My point is simply that this strand of the Newtonian tradition is central to Kant's thinking.

27. See, for example, the very sympathetic account in Whiteside [120], especially §§V, X, XI, making extensive use of the concept of "limit-motions."

28. See the Scholium following Lemma XI of *Principia*, Book I, §1, for example, where Newton appeals to the intuitive idea of instantaneous velocity to justify the existence of the required limits: [82], pp. 87–88; [83], pp. 38–39.

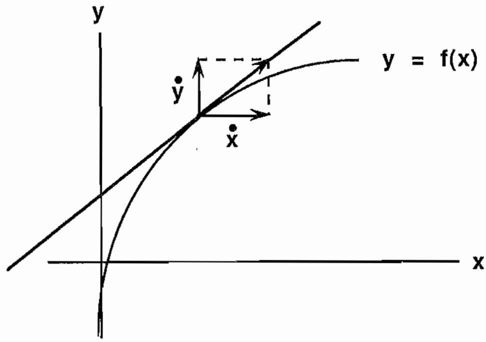

Figure 4

Whiteside [120], pp. 374–376). So the basic notions of the calculus are explained without ever appealing to differentials or infinitely small quantities.

In any case, it is extremely likely that some such understanding of the calculus (the method of fluxions) underlies Kant's insistence on the kinematic character of construction in pure intuition.[29] When he speaks of the "productive synthesis" involved in the "mathematics of extension" Kant is referring to what we would now call calculus in a Euclidean space; he is not simply thinking of Euclidean geometry proper. And, if this is correct, we can better understand why, given the way concepts such as continuity and passage to the limit are understood, there is no possibility of a distinction between pure and applied—uninterpreted versus interpreted—mathematics in the modern sense. The only way one can represent continuity, for example, is to provide what we would now call an intuitive interpretation of the continuity or completeness axiom, an interpretation that necessarily makes that axiom true. In particular, since convergence

29. As far as I know, Kant comes closest to making this explicit in the course of an exchange with August Rehberg in 1790 concerning the nature of irrational numbers and their relation to intuition. In a draft (Reflexion 13) of his reply to Rehberg Kant says: "If we did not have concepts of space then the quantity $\sqrt{2}$ would have no meaning [Bedeutung] for us, for one could then represent every number as an aggregate [Menge] of indivisible units. But if one represents a line as generated through fluxion [durch fluxion], and thus generated in time, in which we represent nothing simple, then we can think 1/10, 1/100, etc., etc. of the given unit" (14, 53.2–7; see also Adickes's note to this passage on pp. 53–54). The entire exchange (Rehberg's letter: 12, 375–377; Kant's drafts (Reflexionen 13–14): 14, 53–59; Kant's reply: 11, 195–199) sheds much light on Kant's view of the relationship between spatio-temporal intuition and arithmetical-algebraic concepts, and calls for a detailed investigation: for a beginning, see Chapter 2 below. Some aspects of the exchange are discussed by Parsons [94]. (I am indebted to Parsons for first calling my attention to the passage at 14, 53.2–7.)

is represented by a *continuous* process of temporal generation, the relevant limit point is automatically generated as well.

Moreover, although the kinematic interpretation of the calculus certainly does not meet modern standards of rigor, it is also not afflicted with the obvious problems about consistency and coherence facing an interpretation based on differentials, infinitesimals, and infinitely small quantities. Indeed, when the kinematic interpretation was explicitly criticized by mathematicians such as D'Alembert and l'Huilier in the late eighteenth century this was not on grounds of coherence and consistency, but rather because it was thought to import a "foreign" or "physical" element into pure mathematics. Pure mathematics should be independent of and prior to mathematical physics; therefore, it should be developed in complete independence of the idea of motion.[30] For Kant, on the other hand, this "mixing" of physical and mathematical ideas is not a defect but a virtue. Since part of the "general doctrine of motion"—namely, pure kinematics or "phoronomy"—is, in effect, also a branch of pure mathematics, it is possible to hold that this part of mathematical physics is a priori as well.[31] So an explicit "mixing" of physical and mathematical ideas is essential to the unity of Kant's system.[32]

30. See the excellent account in Grabiner [39] and, in particular, the quotation from Bolzano's "Rein analytischer Beweis . . . " (1817), §II: "the concept of Time and even more that of Motion are . . . foreign to general mathematics," on p. 53.

31. See §§X–XV of Kästner's analysis text, entitled "Bewegung gehört in die Geometrie." Kästner replies to l'Huilier's criticism of the idea of motion by drawing a sharp distinction between kinematics ("phoronomy")—where one considers the motion of mere mathematical points independently of their physical properties (such as mass); and dynamics—where one explicitly considers both the physical constitution of such points and the forces that produce the motion. The former is a branch of pure mathematics and is therefore a priori; the latter is a branch of physics and is therefore a posteriori. In the third edition of his text (1799), Kästner even refers to Kant's *Metaphysical Foundations of Natural Science* (1786) for a justification of this distinction.

32. In this connection, see the important footnote at B155: "Motion of an *object* [*Objekt*] in space does not belong to a pure science, and consequently not to geometry; for, that something is movable cannot be cognized a priori, but only through experience. But motion, as the *describing* [*Beschreibung*] of a space, is a pure act of successive synthesis of the manifold in an outer intuition in general, and belongs not only to geometry, but even to transcendental philosophy." (That the *describing* of a space = the motion of a mathematical point is confirmed by the Observation to Definition 5 in the first chapter or Phoronomy of the *Metaphysical Foundations of Natural Science:* 4, 489.6–11.) Thus Kant does have a distinction between pure and applied geometry—although it is certainly not *our* distinction. In pure geometry we consider figures generated in "empty" space by the motion of mere mathematical points; in applied geometry we consider the actual sensible objects contained "in" this space. That what holds for mere mathematical points in "empty" space holds also for actual sensible objects found "in" this space (that pure mathematics can be applied) can only be established in transcendental philosophy. See also A165–166/B206–207. (Joshua Cohen, Ralf Meerbote, and Manley Thompson have all emphasized the importance of B155n to me.)

At the same time, however, the difference between the iterative infinity involved in Euclidean constructions and the stronger use of infinity involved in limit operations helps to elucidate the sense in which the kinematic interpretation fails to meet modern standards of rigor. In Euclidean geometry we specify the objects of our investigation—circles, straight lines, and any figures constructible from them—by a well-defined iterative or "inductive" procedure. This specification then underlies our iterative, step-by-step method of proof: the substitution of a previously constructed object—a given finite straight line segment, say—as argument in a further constructive operation—the construction of a circle based on this line segment as radius via Postulate 3, for example. By contrast, in the fluxional calculus we have no such specification: no step-by-step procedure (nor any other precisely defined method) for constructing all fluents or "fließende Größen" has been given.[33] Similarly, our temporal representation of the limit operation does not proceed by repeated application of previously given functions: each new limit has to be constructed "on the spot," as it were. This, in the end, is perhaps the most fundamental advantage of the Cauchy-Bolzano-Weierstrass definition of convergence. For our use of the logical form $\forall\exists\forall$ in an appropriate formal system of quantificational logic permits us to reestablish iterative methods of proof: we can "handle" the points generated via limit operations by rigorous—finitary—deductions.[34]

Be this as it may, the class of curves generated as fluents or "fließende Größen" proves to be inadequate to the needs of mathematics and mathematical physics. The main problem is that, since continuity is explained by continuous motion, *continuity* automatically implies *differentiability* as well (see Whiteside [120], p. 349). A curve generated by continuous motion (drawn by the continuous motion of a pencil, as it were) automatically has a tangent or direction of motion at each point. So the class of continuous curves is assimilated to the class of what we now call *smooth* (differentiable) curves.[35] Actually, this is not quite right, for those who employed the method of fluxions of course knew that there are continuous

33. We might conceive "fluents" as smooth (or at least piece-wise smooth) maps from the real numbers ("time") into some smooth manifold (Euclidean three-space, for example). The point, of course, is that precisely these concepts are unavailable to Newton and Kant.

34. This is perhaps part of what Russell had in mind when he praised the "infinitary" power of quantification in 1903: "An infinitely complex concept, although there may be such, can certainly not be manipulated by the human intelligence; but infinite collections . . . can be manipulated without introducing any concepts of infinite complexity" ([102], §72).

35. Thus Kant replies to a question due to Kästner in the *Inaugural Dissertation*, §14.4, by proving that "the continuous motion of a point over all the sides of a triangle is impossible" (4, 400.4–5). Kant's proof consists in the observation that no tangent or direction of motion exists at any vertex of the triangle. From a modern point of view, then, he "assumes" that a continuous map from **R** ("time") into space is also a smooth map.

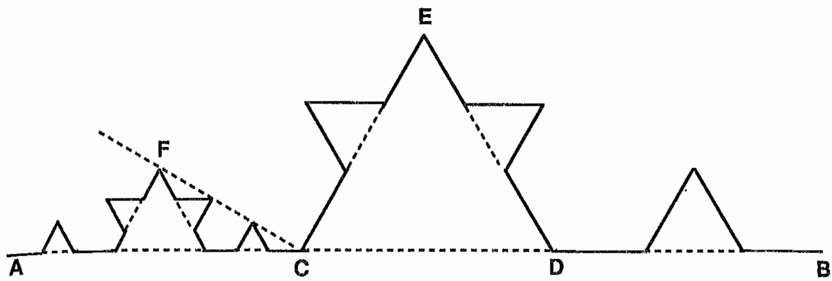

Figure 5

curves that lack tangents at certain points: curves with "cusps" or "corners." However, such curves can be easily comprehended within a kinematic understanding of continuity so long as one can think of them as "pieced together" by a finite number of smooth curves, so long as they have a finite number of "isolated" singular points.

But what happens if we allow such a process of "piecing together" smooth curves to be itself iterated indefinitely: if we apply limit operations to infinite collections of given smooth or "well-behaved" curves? What we get, of course, includes continuous but *nowhere* differentiable curves. The most famous examples of such curves, given by Weierstrass in 1872, are constructed via trigonometric series.[36] Fortunately, however, we can also give simpler, more intuitive examples. Perhaps the simplest is the Koch curve: we start with a horizontal line segment *AB* which we divide into three equal parts by points *C* and *D;* on the middle segment *CD* we construct an equilateral triangle *CED* and erase the open segment *CD;* we repeat the same construction on each of the segments *AC, CE, ED, DB;* finally, we continue this process indefinitely on each remaining segment (see Figure 5). The resulting curve is continuous, but at no point is there a well-defined tangent.[37] (In *every* neighborhood of *C,* for example, both lines *AC* and *FC* intersect infinitely many other points of the curve.) Thus, no finite segment of the Koch curve can be drawn by the continuous motion of a pencil: we must think of each point as laid down independently, as it were, yet nevertheless in a continuous order.

36. See, for example, Boyer [9], pp. 284–285. (As Boyer points out, the first example of a continuous but nowhere differentiable function was given by Bolzano in 1834: [9], pp. 269–270). We should also note that such "pathological" functions arise naturally out of Fourier's work in the 1820s on partial differential equations, work that of course is directly inspired by, and has extremely important applications to, problems in mathematical physics. So the difficulty is in no way confined to pure mathematics.

37. This example was published by H. von Koch in papers of 1903 and 1906. See Eves [29], §13.4. As Eves points out, although the Koch curve is not single-valued, we can easily construct similar single-valued examples.

Such continuous but nowhere differentiable curves clearly exceed the scope of the kinematic interpretation: we cannot understand their continuity via the intuitive idea of continuous motion.[38] To get a mathematical grip on this wider class of curves we need a clear distinction between continuity and differentiability; and this, of course, is one of the main achievements of our modern approach to convergence. We define the *continuity* of a function $f(x)$ at a given point x_0 by an expression of the form $\forall \exists \forall$ [Conv(s, x_0) & Φ], where s' is a sequence defined algebraically from $f(x)$ and Φ says that s converges *to* $f(x_0)$. We define the *differentiability* of $f(x)$ at a given point x_0 by an expression of the form $\forall \exists \forall$ [Conv(s', x_0], where s' is a second sequence defined algebraically from $f(x)$. By understanding these notions formally rather than intuitively, we can, for the first time, both clearly and precisely distinguish them and clearly and precisely explore their logical relations: differentiability logically implies continuity but not vice versa, for example.[39]

· IV ·

The present approach to Kant's theory of geometry follows Russell in assuming that construction in pure intuition is primarily intended to explain mathematical proof or reasoning, a type of reasoning which is therefore distinct from logical or analytic reasoning. Again following Russell, we have sought an explanation for this idea in the difference between the essentially monadic logic available to Kant and the polyadic logic of modern quantification theory. Further, we have tried to link this conception of mathematical reasoning with the very possibility of thinking or representing mathematical concepts and propositions. Thus, for example, "I cannot think a line except by *drawing* it in thought" (B154), because only

38. Kitcher [59] stresses the importance of this problem for Kant's conception of pure intuition, a problem that is perhaps even more fundamental than the discovery of non-Euclidean geometries. As Kitcher remarks: "The death blow was not struck by Bolyai, Lobachevski, and Klein but by the men in the tradition which led to Weierstrass's function, continuous everywhere but differentiable nowhere" ([59], p. 41).

39. It is also worth noting that, although the distinction between continuity and differentiability obviously makes an essential (and rather strong) use of polyadic logic, it is not itself a purely logical distinction: the two formulas have the same logical form, they differ only algebraically. However, one finds precisely such a logical distinction in the contrast between *point-wise* and *uniform* properties. For example, the distinction between point-wise and uniform convergence is purely logical: it is a distinction in quantifier order alone. And this distinction, which is at least obscured in the work of Cauchy, is developed with great subtlety and precision by Weierstrass. In Weierstrass's work, we might say, polyadic quantification theory comes fully into its own. It is perhaps no accident, then, that Frege, who was of course intimately acquainted with Weierstrass's foundational contributions, invented the first accurate and complete formulation of quantificational logic in 1879. (In this connection, see the description of Frege's "advanced course on *Begriffsschrift*" in Carnap [17], p. 6.)

this representation permits me to use the concept of line in mathematical reasoning (such as Euclid's or Newton's) where properties like denseness and continuity play an essential role.

Yet Russell's assumption has been vigorously debated. It has been maintained that Kant did not deny, and indeed may have even affirmed, that mathematical inference is logical or analytic; his primary concern, rather, is with the status of the premises or axioms of such inferences. Geometry is synthetic precisely because its underlying axioms are synthetic; the (synthetic) theorems of geometry then follow purely logically or analytically. This anti-Russellian view is clearly and forcefully stated by Beck:

> The real dispute between Kant and his critics is not whether the theorems are analytic in the sense of being strictly [logically] deducible, and not whether they should be called analytic now when it is admitted that they are deducible from definitions, but whether there are any primitive propositions which are synthetic and intuitive. Kant is arguing that the axioms cannot be analytic . . . because they must establish a connection that can be exhibited in intuition.[40]

As Beck indicates, this view is attractive because Kant will not be refuted, as Russell thought, by the mere invention of polyadic logic. For even modern formulations of Euclidean geometry such as Hilbert's will contain primitive propositions or axioms, and pure intuition can be called in to secure their truth (to provide a model, as it were).[41]

Indeed, from this point of view the discovery of logically consistent systems of non-Euclidean geometry should be seen as a vindication of Kant's conception. The existence of such geometries shows conclusively that Euclid's axioms are not analytic and, therefore, that no analysis of the basic concepts of geometry could possibly explain their truth (as Leibniz apparently thought). Assuming that Euclid's axioms are true, then, there is no alternative but to appeal to a synthetic source: hence pure intuition.[42]

40. Beck, "Can Kant's Synthetic Judgements Be Made Analytic?" (1955), reprinted in [4], pp. 89–90. Martin takes this point of view to extremes, viewing Kant as a forerunner of "modern axiomatics": see [73], for example. Needless to say, such a conception is completely antithetical to the present interpretation.

41. Thus Hilbert, in his brief Introduction to [48], refers to Kant and equates the task of axiomatizing Euclidean geometry with "the logical analysis of our spatial intuition."

42. In the context of contemporary discussion this view has been articulated most clearly and explicitly by Brittan [10]. Indeed, after referring to A220–221/B268 ("there is no contradiction in the concept of a figure which is enclosed within two straight lines, since the concepts of two straight lines and of their coming together contain no negation of figure"), Brittan says: "It was Kant's appreciation of the fact that non-Euclidean geometries are consistent (possibly something of which his correspondent, the mathematician J. H. Lambert, made him aware) that, among several different considerations, led him to say that Euclidean geometry is synthetic. The further development of non-Euclidean geometries only confirms his view" ([10], p. 70, n. 4; Brittan follows Martin [73], §2, in this estimate of the Lambert-

On the Russell-inspired interpretation developed here, by contrast, there can be no question of non-Euclidean geometries for Kant. Non-Euclidean straight lines, if such were possible, would have to possess at least the order properties—denseness and continuity—common to all lines, straight or curved. And, on the present interpretation, the only way to represent (the order properties of) a line—straight or curved—is by drawing or generating it in the space (and time) of pure intuition. But this space, for Kant, is necessarily Euclidean (on both interpretations). It follows that there is no way to draw, and thus no way to represent, a non-Euclidean straight line, and the very idea of a non-Euclidean geometry is quite impossible.[43] (Another way to see the point is to note that the anti-Russellian interpretation would reinstate precisely the modern distinction between pure and applied geometry argued above to be unavailable to Kant.)

The anti-Russellian interpretation draws its primary support from B14:

> For as it was found that all mathematical inferences proceed in accordance with the principle of contradiction [nach dem Satze des Widerspruchs fortgehen] (which the nature of all apodictic certainty requires), it was supposed that the fundamental propositions [Grundsätze] could also be recognized from that principle [aus dem Satze des Widerspruchs erkannt würden]. This is erroneous. For a synthetic proposition can indeed be comprehended [eingesehen] in accordance with the principle of contradiction, but only if another synthetic proposition is presupposed from which it can be derived [gefolgert], and never in itself.

Kant seems to be saying that because inference from axioms to theorems was (correctly) seen as analytic, the axioms themselves were (incorrectly) thought to be analytic. But these axioms are really synthetic; for this reason (and only for this reason), so are the theorems. Kant therefore agrees with Russell that the conditional, Axioms → Theorems, is a logical

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Kant connection). The idea, apparently, is that two-sided plane figures exist in *elliptic* (positive curvature) space—more precisely, in the subcase of *spherical* space, and Kant is supposed to have learned of this type of non-Euclidean space from Lambert. This idea is most implausible, however, for Lambert of course proved that elliptic space is *impossible;* in that elliptic space—unlike hyperbolic (negative curvature, Bolyai-Lobachevsky) space—does contradict the remainder of Euclid's axioms, in particular, the assumed infinite extendability of straight lines (Postulate 2): see, for example, Bonola [8], §§18–22. (Brittan develops an alternative reading of A220–221/B268 and the Lambert-Kant connection in [11], pp. 74–79.)

43. Compare Beth [6], p. 364: "If one assumes this view of geometrical demonstration [that intuition plays an essential role], then *absolutely nothing follows* from the formal possibility of a non-Euclidean geometry, that is, from the formal independence of the Parallel Postulate relative to the remaining axioms of Euclidean geometry. For if we attempt to answer any geometrical question on the basis of the remaining axioms, we must (according to Kant) first construct the corresponding figure. This construction will proceed according to the antecedent laws of pure intuition, and therefore the Euclidean answer will come out at the end. The distinction between axioms and theorems will therefore obviously collapse."

or analytic truth;[44] his point is simply that the antecedent of the conditional is synthetic.

I do not think this reading of the passage is forced on us. First of all, Kant does not actually say that mathematical inference is analytic, nor that the theorems can be analytically derived. Thus the first sentence may mean only that mathematical proofs necessarily involve logical or analytic steps—and, of course, no logical fallacies.[45] Second, it is assumed that by *fundamental propositions* (Grundsätze) Kant means *axioms,* and this is doubtful. Kant's own technical term for axioms is *Axiomen* (see A163–165/B204–206, A732–733/B760–762), and at A25 he calls the proposition that two sides of a triangle together exceed the third a fundamental proposition (Grundsatz). This latter is of course not an axiom in Euclid, but a basic (and therefore fundamental) theorem (Prop. I.20). So the error Kant is diagnosing here may not be the (really rather ridiculous) mistake of transferring analyticity from inference to premise (axiom), but the more subtle supposition that because logic plays a central role in the proof of basic theorems it is sufficient for securing their truth.

A more fundamental problem for the anti-Russellian reading of B14 is posed by Kant's conception of arithmetic. Kant is supposed to have a more-or-less modern picture of mathematical theories as strict deductive systems. The synthetic character of mathematics depends solely on the synthetic character of the underlying axioms. But this is certainly not Kant's picture of arithmetic. According to Kant, arithmetic differs from geometry precisely in having no axioms, for there are no propositions that are both general and synthetic serving as premises in arithmetical arguments (A163–165/B204–206). Thus our conception of arithmetic as based on the Peano axioms, say, is completely foreign to Kant, and one cannot use the model of an axiomatic system to explain why arithmetic is synthetic: one cannot suppose that arithmetical reasoning proceeds purely logically or analytically from synthetic axioms as premises.[46]

44. But we should remember that the Russell of 1903 still believed that *logic* is synthetic. See [102], §434: "Kant never doubted for a moment that the propositions of logic are analytic, whereas he rightly perceived that those of mathematics are synthetic. It has since appeared that logic is just as synthetic as all other kinds of truth."

45. Compare A59/B84, where the principle of contradiction is said to be a necessary, but insufficient, criterion for *all* truth, and A151/B190, where it is asserted that the truth of *analytic judgements* "can be sufficiently recognized according to the principle of contradiction [nach dem Satze des Widerspruchs hinreichend können erkannt werden]."

46. Of course this difference between arithmetic and geometry is explicitly recognized by Beck: he suggests that Kant's discussion of arithmetic is simply inconsistent with the general account of mathematics at B14 (see [4], p 89). Compare also Brittan [10], pp. 50–51. Martin, on the other hand, goes so far as to attribute an axiomatic conception of arithmetic to Kant: see [73] and especially [74]. Yet Martin relies almost exclusively on the writings of Kant's contemporaries and students—Johann Schultz, in particular—and has no account of Kant's

Yet arithmetic is the very first example Kant uses (at B15–16) to illustrate, and presumably illuminate, the general ideas of B14. Arithmetical propositions such as 7 + 5 = 12 are synthetic, not because they are established by analytic derivation from synthetic axioms (as we would derive them from the Peano axioms, say), but because they are established by the successive addition of unit to unit. This procedure is synthetic, according to Kant, because it is necessarily temporal, involving "the successive progression from one moment to another" (A163/B203).[47] Thus, for example, only the general features of succession and iteration in time can guarantee the existence and uniqueness of the sum of 7 and 5, which, as far as logic and conceptual analysis are concerned, is so far merely possible (non-contradictory). Similarly, only the unboundedness of temporal succession can guarantee the infinity of the number series, and so on.

For Kant, then, arithmetical propositions are established by calculation, a procedure that is sharply distinguished from discursive argument in being essentially temporal. This is why Kant says that the synthetic character of arithmetical propositions "becomes even more evident if we consider larger numbers, for it is then obvious that, however we might turn and twist our concepts, we could never by mere analysis unaided by intuition be able to find the sum"(B16). The reference to larger numbers makes it clear that intuition is not being called in to secure the truth of basic propositions—such as 2 + 2 = 4, perhaps—by "setting them before our eyes." Rather, intuition underlies the *step by step* process of calculation which, in its entirety, may very well not be surveyable "at a glance."[48]

We have now reached the heart of the matter, I think, for it is the idea of a sharp distinction between calculation and discursive argument that is perhaps most basic to Kant's conception of the role of intuition in mathematics. Thus, at A734–736/B762–764 Kant contrasts mathematical and philosophical reasoning. Only mathematical proofs are properly called *demonstrations,* while philosophy is restricted to conceptual or discursive ("acroamatic") proofs. The latter "must always consider the uni-

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explicit assertion—repeated in the letter to Schultz of November 25, 1788—that arithmetic has no axioms. For this reason he is justly criticized by Parsons, "Kant's Philosophy of Arithmetic," §III, who rightly points out that what is most striking here is the *difference* between Kant and Schultz.

47. Compare A162/B202–203, where a "synthesis of the manifold whereby the representation of a determinate space or time is generated, that is through the combination of the homogeneous [Zusammensetzung des Gleichartigen]" is said to underly all concepts of magnitude, and A143/B182: "Number is therefore simply the unity of the synthesis of the manifold of a homogeneous intuition in general, a unity due to my generating time itself in the apprehension of the intuition." See Parsons, "Philosophy of Arithmetic," §§VI, VII, for a rich and penetrating discussion of such passages.

48. See Young [124] for a very interesting and helpful discussion of the role of calculation in Kant's conception of arithmetic.

versal *in abstracto* (by means of concepts)," the former "can consider the universal *in concreto* (in the single intuition), and yet still through pure a priori representation whereby all errors are at once made visible [sichtbar]" (A735/B763).

That Kant has calculation centrally in mind here is indicated by his reference to the methods employed in solving algebraic equations (A734/B762), a reference which recalls the even more explicit conception of calculation found in the *Enquiry Concerning the Clarity of the Principles of Natural Theology and Ethics* (1764). See, for example, the First Reflection, §2, entitled "Mathematics in its methods of solution [Ausflösungen], proofs, and deductions [Folgerungen] examines the universal under symbols *in concreto;* philosophy examines the universal through symbols *in abstracto*":

> I appeal first of all to arithmetic, both the general arithmetic of indeterminate magnitudes [algebra], as well as that of numbers, where the relation of magnitude to unity is determinate. In both symbols are first of all supposed, instead of the things themselves, together with special notations [Bezeichnungen] for their increase and decrease, their ratios, etc. Afterwards, one proceeds with these signs, according to easy and secure rules, by means of substitution, combination or subtraction, and many kinds of transformations, so that the things symbolized are here completely ignored, until, at the end, the meaning of the symbolic deduction is finally deciphered [entziffert]. (2, 278.12–26)

As the Third Reflection, §1, explains, this "symbolic concreteness" of mathematical proof accounts for the difference between philosophical and mathematical certainty. Since philosophical argument is discursive or conceptual, ambiguities and equivocations in the meanings of general concepts are always possible. Mathematics, on the other hand, works with concrete or singular representations that allow us to be assured of the correctness of its substitutions and transformations "with the same confidence with which one is assured of what one sees before one's eyes" (291.29–30). As Kant puts it in the first *Critique,* the step by step application of the easy and secure rules of calculation "secures all inferences against error by setting each one before our eyes" (A734/B762).[49]

From the present point of view, the point could perhaps be recon-

49. These passages, and the closely related passage at A715–718/B743–746, are illuminatingly discussed by Parsons, "Philosophy of Arithmetic," and especially by Thompson [108], who distinguishes between "diagrammatic" and discursive proofs. Beth [6] was perhaps the first, in the context of contemporary discussion, to emphasize the importance of these passages for understanding Kant's philosophy of mathematics. The sharp distinction Kant draws between discursive ("acroamatic") proof and mathematical proof (demonstration) seems to me to establish *part* of the Russellian assumption beyond the shadow of a doubt: mathematical reasoning *cannot* be purely logical for Kant. What has still to be established is that the inferential use of pure intuition is *primary.*

structed as follows. Mathematical proof, unlike discursive proof, operates not only with *predicates* such as ⌜x is even⌝ and ⌜x is a triangle⌝, but first and foremost with *function-signs* such as ⌜$x + y$⌝ and ⌜the bisector of z⌝. In calculation we form functional terms by inserting particular arguments into the function-signs, we set up equalities (and inequalities) between such functional terms, and we substitute one functional term for another in accordance with these equalities. Since both the arguments and the values of our function-signs are individuals,[50] the procedure of *substitution* is to be sharply distinguished from the *subsumption* of individuals under general concepts characteristic of discursive reasoning. In particular, the essence of the former procedure lies in its iterability: $f(a)$ can be substituted in $f(x)$ to form a distinct functional term $f(f(a))$, while it of course makes no sense at all to subsume the predication $F(a)$ under the predicate $F(x)$.[51] Thus, the essentially "extra-logical" form of inference required is that which takes us from one object a satisfying a condition . . . a . . . to a second object $f(a)$ satisfying another condition ___$f(a)$___, and from there to a third object $f(f(a))$ satisfying ---$f(f(a))$---, and so on.

Now this conception of the role of calculation and substitution in mathematical proof also applies, *mutatis mutandis,* to the case of geometry. In Euclidean geometry we start with an initial set of basic constructive functions: the operation $f_L(x, y)$ taking two points x, y to the line segment between them, the operation $f_E(x, y)$ taking line segments x, y to the extended line segment of length $x + y$, and the operation $f_C(x, y)$ taking point x and line segment y to the circle with center x and radius equal to y. We also have a specifically geometrical equality relation (congruence) and, of course, definitions of the basic geometrical figures (circle, triangle, and so on). Euclidean proof then proceeds somewhat as follows. Given a figure a satisfying a condition . . . a . . . , we construct, by iteration of the basic operations, a new constructive function g yielding an expanded figure $g(a)$ satisfying a condition ---$g(a)$---. From this last proposition we are then able to derive a new condition ___a___ on our original figure a.

50. See also Hintikka, "Kant on the Mathematical Method," §6, for an illuminating discussion of the role of function-signs in Kant's conception of algebraic construction.

51. Compare the conception in the *Tractatus* [123] of mathematics as based on "calculation" and "operations" at 6.2–6.241, along with the distinction between "operations" and "[propositional] functions" at 5.251: "A [propositional] function cannot be its own argument, whereas an operation can take one of its own results as its base." Just as in the *Tractatus,* however, it is hard to see how such a "calculational" conception can yield more than primitive recursive arithmetic: see Thompson [108], n. 21 on p. 341. The essential difference between Kant and the *Tractatus* here is that Wittgenstein also applies the notion of "iterative operations" to *logic* in "the general form of a proposition" (6), whereas Kant uses this notion *to distinguish* logic and mathematics. This, of course, is because Wittgenstein is operating in the context of Frege's much stronger logic, where iterative construction of *propositions* via truth-functions and quantifiers plays a central role.

Whereas the inference from ---*g*(*a*)--- to . . . *a* . . . can be viewed as "essentially monadic," and is therefore analytic or logical for Kant, the inference from . . . *a* . . . to ---*g*(*a*)--- is not: it proceeds synthetically, by expanding the figure *a* as far as need be into the space around it, as it were. Since this procedure is grounded in the indefinite iterability of our basic constructive operations, geometry is synthetic for much the same reasons as is arithmetic; and, therefore, the case of arithmetic is primary.[52]

Confirmation is apparently provided by the discussion at B15–17. For Kant illustrates B14 at great length with the example of arithmetic and only then touches on geometry, almost as a corollary:

> Just as little is any fundamental proposition [Grundsatz] of geometry analytic. That the straight line between two points is the shortest is a synthetic proposition. For my concept of *straight* [*Geraden*] contains nothing of quantity [Größe], but only a quality. The concept of shortest is entirely an addition, and cannot be derived by any analysis of the concept of straight line. The aid of intuition must therefore be brought in, by means of which alone the synthesis is possible. (B16–17)[53]

As the discussion of arithmetic has shown, the general concept of magnitude [Größe] requires an intuitive synthesis (the successive addition of unit to unit). But geometry requires this concept as well (for example, in connecting the notion of straight line with the notion of shortest line). Therefore, geometry, just as much as arithmetic, is a synthetic discipline.[54]

Nevertheless, there is of course an important difference between the two cases. As already noted above, geometry has axioms whereas arithme-

52. Thus I cannot follow Parsons when he draws a sharp distinction between the cases of arithmetic and geometry, and even endorses an interpretation of the geometrical case of the Beck-Brittan variety (see "Philosophy of Arithmetic," §§II, IV, and p. 128 of [92]). On the contrary, I think Kant's views can only be understood if we apply the ideas Parsons has developed for the case of arithmetic to the case of geometry also.

53. See also the *Enquiry,* I, §2, where the discussion follows the same order, and the letter to Johann Schultz of November 25, 1788, where the priority of arithmetic is stated rather explicitly: "General arithmetic (algebra) is an *ampliative* [sich *erweiternde*] science to such an extent that one cannot name another rational science equal to it in this respect. Indeed, the other parts of pure mathematics [reine Mathesis] await their own growth [Wachstum] largely from the amplification [*Erweiterung*] of this general theory of magnitude" (10, 555.10–14).

54. It is interesting to speculate on what exactly Kant has in mind in his example of the geodesicity of straight lines. This proposition appears as neither an axiom nor a theorem in Euclid, but it was stated as an *assumption* by Archimedes (Heath [46], pp. 166–169). Kant does not appear to endorse this idea, and it is perhaps most plausible to suppose that he is referring to the variational methods developed by Euler in 1728 for *proving* geodesicity. That is, we consider the result of integrating arc-length over all possible (neighboring) curves joining two given points, and we look for the curve that minimizes the integral. Here, of course, the idea of "synthesis" (in the guise of integration) is especially prominent. But this is so far just speculation.

tic does not; moreover, geometry uses "ostensive construction (of the objects [Gegenstände] themselves)" (A717/B745) in addition to the "symbolic" or "characteristic" (A734/B762) construction common to algebra and arithmetic. Thus Kant's discussion of algebraic construction has a decidedly "formalistic" tone: we "abstract completely from the nature of the object" (A717/B745) and, as the above passage from the *Enquiry* puts it, "symbols are first of all supposed, instead of the things themselves" and "the things symbolized are here completely ignored." In geometry, on the other hand, such "formalism" is quite inappropriate: geometrical construction operates with "the objects themselves" (lines, circles, and so on).

This difference between arithmetical-algebraic construction and geometrical construction is perhaps most responsible for the confusion that has surrounded Kant's theory. For it begins to look as if geometrical intuition has not merely an inferential or calculational role, but also the more substantive role of providing a model, as it were, for one particular axiom system as opposed to others (Euclidean as opposed to non-Euclidean geometry). Intuition does this, presumably, by placing the objects themselves before our eyes, whereby their specific (Euclidean) structure can be somehow discerned.

From the present point of view, of course, there can be no question of picking out Euclidean geometry from a wider class of possible geometries. Rather, the difference between geometrical and arithmetical-algebraic construction is understood as follows. Geometry, unlike arithmetic and algebra, operates with an initial set of specifically geometrical functions (the operations f_L, f_E, and f_C) and a specifically geometrical equality relation (congruence). To do geometry, therefore, we require not only the general capacity to operate with functional terms via substitution and iteration (composition), we also need to be "given" certain initial operations: that is, intuition assures us of the existence and uniqueness of the values of these operations for any given arguments. Thus the axioms of Euclidean geometry tell us, for example, "that between two points there is only one straight line, that from a given point on a plane surface a circle can be described with a given straight line" (*Inaugural Dissertation,* §15.C: 2, 402.33–34), and they also link the specifically geometrical notion of equality (congruence) with the intuitive notion of superposition (*Prolegomena,* §12: 4, 284.22–26).[55]

55. Serious complications stand in the way of the full realization of this attractive picture. First, of course, Euclid's Postulate 5, the Parallel Postulate, does not have the same status as the other Postulates: it does not simply "present" us with an elementary constructive function which can then be iterated (thus, given two straight lines falling on a third with interior angles on one side together less than two right angles, Postulate 5 not only tells us that we can extend these lines *ad infinitum* on this side, but also says what will happen in

Now one might at first suppose that the case of arithmetic is precisely the same. After all, we need intuitive assurance that the successor function, say, is uniquely defined for all arguments. But the point, I think, is that the successor function is not a specific function at all for Kant; rather, it expresses the general form of succession or iteration common to all functional operations whatsoever. So it is not necessary *to postulate* any specific initial functions in arithmetic: whatever initial functions there may be, the existence and well-definedness of the successor function is guaranteed by the mere form of iteration in general (that is, time). Thus, in explaining why geometry has axioms while arithmetic does not at A164–165/B205–206, Kant refers to the need in geometry for general "functions of the productive imagination" such as our ability to construct a triangle from any three line segments such that two together exceed the third (this functional operation is of course definable, in Euclidean geometry, from the operations f_L, f_E, and f_C: Prop. I.22). The point, presumably, is that no such specific functional operations need be postulated in arithmetic.[56]

In any case, the idea that pure intuition plays the more substantive role of providing a model for one particular axiom system as opposed to others—as the anti-Russellian interpretation requires—is rather obviously untenable and definitely unKantian. The untenability of such a view has been clearly brought out in an instructive article by Kitcher [59].[57] Kitcher supposes that the primary role of pure intuition is to discern the metric and projective properties (the Euclidean structure) of space. We construct geometrical figures like triangles and somehow "see" that they are Euclidean: "[Kant's] picture presents the mind bringing forth its own creations

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the limit: the two lines must meet, and not simply approach one another asymptotically). Second, as argued in §III, geometry for Kant includes the new calculus: the method of fluxions. And this calculus, unlike Euclidean geometry proper, has no basis at all in a finite set of initial constructive functions. So intuition has the even more substantive role of creating each new object "on the spot," as it were. These two complications reflect deep mathematical problems that are only fully solved in the next century through the discovery of non-Euclidean geometries and the independence of the Parallel Postulate, on the one hand, and through the "rigorization" and eventual "arithmetization" of analysis, on the other. In the end, therefore, the relation between arithmetic and geometry remains a source of fundamental, and unresolved, tensions in Kant's philosophy. Yet it is surely remarkable that these tensions arise precisely in connection with some of the deepest mathematical questions of the time.

56. These ideas can once again profitably be compared with *Tractatus* 6.01–6.031: "Number is the exponent of an operation." For a fuller discussion of arithmetical-algebraic construction see Chapter 2 below.

57. Kitcher himself remains officially neutral on the issue between Beck and Russell. He suggests ([59], §IV) that pure intuition may play a role in proofs, and even makes some interesting remarks about the use of pure intuition in (Kant's conception of) Newton's fluxional reasoning ([59], p. 41). Nevertheless, Kitcher's setting of the problem only makes sense in the context of an interpretation of the Beck-Brittan variety.

and the naive eye of the mind scanning these creations and detecting their properties with absolute accuracy" ([59], p. 50). It is then easy to show that pure intuition, conceived on this quasi-perceptual model, could not possibly perform such a role. Our capacity for visualizing figures has neither the generality nor the precision to make the required distinctions.

Thus, for example, Kant's appeal to the proposition that two sides of a triangle together exceed the third at A25/B39 is considered, and "[w]e now imagine ourselves coming to know [it] in the way Kant suggests. We draw a scalene triangle and see that this triangle has the side-sum property" ([59], p. 44). But this idea quickly founders on Berkeley's generality problem: how are we supposed to conclude that all triangles have the side-sum property and not, say, that all triangles are scalene? (Actually, in this connection a more relevant dimension of generality is *size*. In elliptic—positive curvature—space, triangles that are "small"—relative to the dimensions of the space itself—have the side-sum property while arbitrarily large triangles do not. So one cannot argue from the properties of small, visualizable triangles to the properties of arbitrary triangles.)[58]

It is extremely unlikely, however, that in appealing to intuition at A25/B39 Kant is imagining any such process of "visual inspection." It is much more plausible that, in precise parallel to his discussion of the angle-sum property at A715–717/B743–745, he is referring to the Euclidean *proof* of this proposition (Prop. I.20). We consider a triangle ABC and prolong BA to point D such that DA is equal to CA (see Figure 6). We then draw DC, and it follows that $\measuredangle ADC = \measuredangle ACD$ and therefore $\measuredangle BCD > \measuredangle ADC$. Since the greater angle is subtended by the greater side (Prop. I.19), $DB > BC$. But $DB = BA + AC$; therefore $BA + AC > BC$. Q.E.D. Intuition is required, then, not to enable us to "read off" the side-sum property from the particular figure ABC, but to guarantee that we can in fact prolong BA to D by Postulate 2.[59] (It is precisely this Postulate that fails

58. Closely related considerations are presented by Hopkins in his penetrating criticism of Strawson's attempted reconstruction [54]. The basic point is that "in the small," wherein alone "visual inspection" is possible, Euclidean and non-Euclidean geometries are quite indistinguishable.

59. In Kant's technical terminology, Berkeley's generality problem is solved via the distinction between *images* and *schemata*—only the latter figure essentially in geometrical proof: "No image [Bild] could ever be adequate to the general concept of triangle. For it would never attain the generality of the concept which makes it valid for all triangles—whether right-angled, obtuse-angled, etc.—but is always limited to a part of this sphere. The schema of a triangle can exist nowhere but in thought and signifies [bedeutet] a rule of synthesis of the imagination in respect to pure figures [Gestalten] in space (A141/B180). (Ralf Meerbote emphasized the importance of this passage to me.) As A164–165/B205 makes clear, this "rule of synthesis" is nothing but the Euclidean construction of a triangle from any three line segments such that two together exceed the third of Proposition I.22.

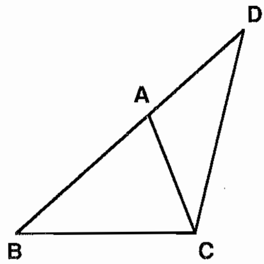

Figure 6

in elliptic space: straight lines are finite—but unbounded—and cannot be indefinitely prolonged at will.)

Now, as Heath observes in his commentary on Proposition I.20, "It was the habit of the Epicureans . . . to ridicule this theorem as being evident even to an ass and requiring no proof" ([46], p. 287), and one might be tempted to suppose that Kant holds a similar view (so that the "visual inspection" metaphor is appropriate after all). This suggestion is immediately quashed by the important discussion of mathematical method at Bxi–xii, however. Kant applauds Diogenes Laertius for naming "the reputed author of even the least important elements of geometrical demonstrations, even of those which, according to ordinary judgement, require absolutely no proof," and concludes:

> A new light dawned on the first man (whether he may be Thales or whoever) who demonstrated the *isosceles triangle;* for he found that he must not inspect what he saw in the figure [Figur], or even in the mere concept of it, and as it were learn its properties therefrom, but he must rather bring forth what he himself has injected in thought [hineindachte] and presented (through construction) according to concepts, and that, in order to know something a priori, he must attribute nothing to the thing except that which follows necessarily from what he himself has placed in it in accordance with his concept.

Kant's example here, of course, is the discovery (sometimes attributed to Thales) of the Euclidean proof that the angles at the base of an isosceles triangle are equal (Prop. I.5),[60] a proof which also proceeds by means of

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We might represent it, then, by a constructive function $f_T(x,y,z)$ which, as Proposition I.22 shows, is definable from the basic constructive functions f_L, f_E, and f_C. Kant's point is simply that to do geometry we need such (general!) constructive functions (to represent our "existence assumptions"). So we do not establish geometrical propositions by "inspection" of the resulting images, but by rigorous proof from axioms and definitions.

60. The reference to Proposition I.5 is made explicit in a letter to Christian Schütz of June

an ingenious expansion of our original triangle into several additional triangles via "auxiliary construction." I do not see how there can be any doubt, therefore, that Kant's "new method" of geometry is precisely Euclid's procedure of construction with straight-edge and compass.[61]

Once again, we are forced to conclude that the primary role of pure intuition is to underwrite the constructive procedures used in mathematical proofs. Moreover, when Kant himself uses "visual inspection" and "eye of the mind" metaphors, it is almost always in connection with inference and proof. Thus, in the passage from the *Enquiry,* Third Reflection, §1, quoted above, Kant says that we can check the correctness of algebraic substitutions and transformations "with the same confidence with which one is assured of what one sees before one's eyes." At A734/B762 we are told that the procedure of algebra "secures all inferences against error by setting each one before our eyes." The intuition involved here is not a quasi-perceptual faculty by which we "read off" the properties of triangles from particular figures, but that involved in checking proofs step by step to see that each rule has been correctly applied: in short, the intuition involved in "operating a calculus." The only apparent exception of which I am aware is *Inaugural Dissertation,* §15.C, where Kant says that some Euclidean axioms (Postulates 1 and 3) are "seen, as it were, in space *in concreto.*" Yet the fact that Kant does not use such language in the first *Critique* suggests that he himself became sensitive to its possible misuse.

Indeed, there is no room in the critical philosophy for the picture underlying the anti-Russellian conception of pure intuition. That conception views Euclidean geometry as a body of truths that selects one structure for space from the much wider class of all possible such structures. Since both Euclidean and non-Euclidean axiom systems are consistent, we need

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25, 1787, where Kant changes "gleichseitiger" in the printed text to "gleichschenkligter": 10, 489.30–32.

61. Hintikka has emphasized the importance of the passage at Bxi–xii (for example, in "Kant's 'New Method of Thought' and His Theory of Mathematics," §2) and the fact that Euclid's proof-procedure provides a model for Kant's notion of construction (especially in "Kant and the Mathematical Method"). Hintikka also rightly emphasizes that Kant's conception of mathematical method is therefore to be sharply *distinguished* from a naive "visual inspection" view. Yet in his zeal to refute Russell's contention that Kant's view of geometry requires an "extra-logical element," Hintikka overlooks the fact that his own reconstruction of the analytic/synthetic distinction (see note 14 above) allows us to do justice to both Russell and Kant: Euclidean constructive proofs do indeed require an "extra-logical element"—if logic, as Kant thought, is syllogistic or (essentially) monadic logic (and this, of course, is precisely Russell's point). See [52], chap. IX, §§4–7, especially p. 218, n. 45, where Hintikka is driven to equate Kant's conception of geometrical reasoning with that of Leibniz and Wolff.

to call on pure intuition to provide a model, as it were, for one system rather than another. As it happens, intuition picks out the Euclidean system.[62] The problem is that Kant has no notion of possibility on which both Euclidean and non-Euclidean geometries are possible. His official notion is "that which agrees with the formal conditions of experience (according to intuition and concepts)" (A218/B265). Mere absence of contradiction is quite insufficient to establish a possibility (A220–221/B267–268); and "To determine [a geometrical figure's] possibility, something more is required, namely, that such a figure be thought under pure [lauter] conditions on which all objects of experience rest" (A224/B271). Accordingly, Kant complains that "the poverty of our customary arguments by which we throw open a great realm of possibility, of which all that is actual (the objects of experience) is only a small part, is patently obvious" and concludes "this alleged process of adding to the possible I refuse to allow. For that which has still to be added would be impossible" (A231/B284).

What produces confusion here is the circumstance that Kant is operating with two notions of possibility: "logical possibility," given by the conditions of thought alone; and "real possibility," given by the conditions of thought plus intuition (compare, for example, Bxxvi,n). One then supposes that the former is a wider genus (containing both Euclidean and non-Euclidean spaces) of which the latter is a species (containing only Euclidean space). But this line of thought employs a notion of *logical possibility* that is completely foreign to Kant. Kant's conception of logic is not that of modern quantification theory, and he can have no notion like ours of all logically possible structures—all models of consistent first-order (or second-order) theories, say. Thus, for example, while there may be no (monadic!) contradiction in the concept of a non-Euclidean figure such as "the concept of a figure which is enclosed within two straight lines" (A220/B268), this does not mean that there is a possible non-Euclidean structure containing such a figure. For a non-Euclidean structure would have to possess the topological properties (denseness and continuity) common to Euclidean and non-Euclidean spaces, and this, for Kant, is impossible. There is only one way even to think such properties: in the space and time of *our* (Euclidean) intuition. Considered independently of *our* sensible intuition, then, the concept of a non-Euclidean

62. This picture is explicit in Kitcher [59], §§I, II, and Brittan [10], chaps. 1–3—both of which make heavy use of contemporary "possible worlds" jargon. The same idea is found in Parsons's more circumspect discussion in "Philosophy of Arithmetic," pp. 117, 128. As Brittan points out ([10], p. 70, n. 4), this picture appears to correspond to Frege's conception of Euclidean geometry: for example, in [33], pp. 20–21. Yet Frege's conception is surely not Kant's, for it is only possible in the context of Frege's much stronger logic.

figure remains "empty" and lacks both "sense and meaning [Sinn und Bedeutung]" (B149).[63]

A closely related point is that pure mathematics is not a body of truths with its own peculiar subject matter for Kant.[64] There are no "mathematical objects" to constitute this subject matter, for the sensible and perceptible objects of the empirical world (that is, "appearances") are the only "objects" there are. For this reason, pure mathematics is not, properly speaking, a body of knowledge (cognition):

> Through the determination of pure intuition we can acquire a priori cognition of objects (in mathematics), but only with respect to their form, as appearances; whether there can be things that must be intuited in this form, is still left undecided. Therefore, mathematical concepts are not in themselves cognitions, except in so far as one presupposes that there are things that can be presented to us only in accordance with the form of this pure sensible intuition. (B147)

Hence, only *applied* mathematics has a subject matter (the sensible and empirical world), and only *applied* mathematics yields a body of substantive truths.[65] Pure mathematics is a mere form of representation (on the present interpretation, a form of reasoning), whose applicability to the chaotic sensible world must be proved by transcendental deduction. In this sense, pure intuition cannot be said to provide a model for Euclidean geometry at all; rather, it provides the one possibility for a rigorous and rational *idea* of space. That there is a model or realization of this idea is not established by pure intuition, but by Kant's own transcendental philosophy.

In the end, therefore, Euclidean geometry, on Kant's conception, is not to be compared with Hilbert's axiomatization, say, but rather with Frege's *Begriffsschrift*.[66] It is not a substantive doctrine, but a form of rational

63. Kant's conception of possibility can then perhaps be explained as follows. Whereas Kant does distinguish between the conditions of *thought* alone and the conditions of *cognition* (thought plus intuition), the former do not correspond to our notion of logical possibility but, rather, to the "empty" idea of the "thing-in-itself." Thus what best approximates our notion of logical possibility is given by the conditions of thought plus *pure* intuition: namely, pure mathematics. "Real possibility," then, is given by the conditions of thought plus *empirical* intuition: namely, (the pure part of) mathematical physics. So "real possibility" most closely corresponds to our notion of "physical possibility."

64. Here I follow Thompson [108], pp. 338–339. See also Parsons [92], pp. 147–149 (Postscript to "Kant's Philosophy of Arithmetic").

65. Of course Kant is not using our *modern* distinction between pure and applied mathematics here: see note 32 above.

66. What I say here actually corresponds more closely to Wittgenstein's conception (in the *Tractatus*) of Frege's *Begriffsschrift* than to Frege's own. Frege's own conception is far less "formalistic." In particular, the laws of logic are in a sense scientific laws like those of any other discipline: it is just that they are maximally general laws (containing variables of

representation: a form of rational argument and inference. Accordingly, its propositions are established, not by quasi-perceptual acquaintance with some particular subject matter, but, as far as possible, by the most rigorous methods of proof—by the proof-procedures of Euclid, Book I, for example. There remains a serious question about Euclid's axioms, of course; when pressed, Kant would most likely claim that they represent the most general conditions under which alone a concept of extended magnitude—and therefore a rigorous conception of an external world—is possible (see A163/B204). And, of course, we now know that Kant is fundamentally mistaken here. In 1854 Riemann developed the general concept of *n-fold extended manifold*—containing three-dimensional Euclidean space as one very special case alongside of more additional possibilities than Kant (or anyone else in the eighteenth century) ever imagined. In 1879 Frege developed a logical framework which makes possible the even more general concept of *relational structure*—under which are subsumed all models for Hilbert's geometry and even, as we now say, all "logically possible worlds." Yet Kant is surely not to be reproached for failing to anticipate the leading logical and mathematical discoveries of a later age; he is rather to be applauded for the depth and tenacity of his insight into the logical and mathematical practice of his own.

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different levels, but no non-logical constants). See the remarkable series of papers by Ricketts, [99], [100], and [101], which depict both Frege's conception of logic and the internal pressures pushing that conception towards the *Tractatus*.

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