thority necessary to make intelligible the bindingness of determinately contentful norms has three dimensions: social, inferential, and historical. In this chapter I have not been able to pursue the intricate interactions among these dimensions that Hegel delineates for us. But I have tried to sketch what I take to be Hegel's most basic thought: his way of working out the Kant-Rousseau insight about a fundamental kind of normativity based on autonomy according to the model of reciprocal authority and responsibility whose paradigm is mutual recognition. I think this is the master idea that animates and structures Hegel's metaphysics and logic.[41]

And as a sort of a bonus, we have also, I hope, seen enough to know how to respond to the puzzle I raised about how to understand Hegel's talk of Spirit as a whole as a self-conscious individual *self*, in the context of his insistence on the irreducibly *social* character of the achievement of self-consciousness. The reciprocal recognitive structure within which Spirit as a whole comes to self-consciousness is *historical*. It is a relation between different time slices of Spirit, in which the present acknowledges the authority of the past, and exercises an authority over it in turn, with the negotiation of their conflicts administered by the future. This is the recognitive structure of *tradition*, which articulates the normative structure of the process of *development* by which concepts acquire their contents by being applied in experience. This process is what Hegel's pragmatism and his idealism aim ultimately to illuminate. Making that structure explicit is achieving the form of self-consciousness Hegel calls "Absolute Knowledge," some of the outlines of which I have tried to convey here.

8

Frege's Technical Concepts

Today we find ourselves at the outset of a golden age in the interpretation of Frege's philosophical writings. Judged by the number of articles, books, and seminars addressing his thought, interest in Frege is at an all-time high. More important, as Frege has come out of the shadow of Russell and Wittgenstein into the full light of critical attention, the degree of sophistication of discussion has achieved a quantum improvement. Many factors conspired to bring about this result, but two events may be singled out as having made special contributions both to the resurgence of interest in and to our greater understanding of Frege's work.

First is the publication, more than sixty years after his death, of that part of his *Nachgelassene Schriften* which survived the vicissitudes of the intervening years. These papers appeared in German in 1969 and in English in 1979.[1] Some of the contents are rough in form, though not without value. We are offered, for example, tables of contents and partial drafts of a textbook on logic and its philosophy which Frege made starts on at various crucial periods of his life. Even draft fragments of this sort permit important inferences from the order of presentation and different emphases given various topics to conclusions about the explanatory priorities Frege associated with his central technical concepts. But not all of the selections represent rough cuts or abandoned projects. Included are some fully polished articles, dealing with Frege's most central technical concepts—fine examples of his concise, sometimes lapidary mathematician's prose—which he had tried unsuccessfully to publish. In a number of cases, these additional texts permit the resolution of exegetical disputes occasioned by what can now be seen to be accidental lacunae and merely apparent emphases in the canonical published corpus.

The other landmark event is the publication in 1973 of Michael Dummett's monumental and long-awaited full-length treatment of Frege's philosophy of language.[2] It would be difficult to overestimate the significance of this classic work. Anyone interested in the interpretation of Frege must give it the same close attention owed to the primary texts. Its clarity of thought, patient rehearsal of considerations, and exercise of the best critical judgment in final appraisal will not be soon equaled. This chapter will not offer a systematic account of Dummett's views, since the most important of these are so intimately tied up with the development of powerful novel approaches to contemporary philosophy of language as to defy brief characterization, even by their author. The original volume has now been supplemented with another containing many valuable amplifications and clarifications.[3] The result is a 1,300-page corpus which, Dummett's complaints[4] to the contrary notwithstanding, by now deserves to be considered as setting out the canonical reading of Frege. It is so considered by the authors discussed below, and forms the background against which their own accounts are set out.

Two examples will serve to indicate the sort of interpretive advance signaled by these events. First, it was widely believed in the 1950s and 1960s that Frege intended the distinction between sense and reference to apply not to functional expressions such as predicates but only to complete expressions such as terms and sentences.[5] Although the famous essay on sense and reference does not discuss such an application of that distinction, the *Nachlass* makes clear that this is only because that discussion was reserved for a further article which is quite explicit in its endorsement of that application, but which was repeatedly rejected for publication until Frege abandoned the attempt. Several other passages reprinted in *Frege: Posthumous Writings* (see note 1) decisively refute the interpretation which would restrict the distinction to complete expressions. A somewhat less important mistake may also be mentioned as indicative, which was done in as much by Dummett's arguments as by the unearthing of further evidence. In "On Sense and Reference" Frege says, "One might also say that judgements are distinctions of parts within truth-values," and that "the reference of the word is part of the reference of the sentence."[6] These remarks have sparked the attribution of a variety of bizarre ontological views to Frege, centering on the notion of the True as representing the whole world, sometimes conceived as a Tractarian world of facts, sometimes as composed of objects (and what about the False?). The remarks stem from a hasty assimilation, soon explicitly rejected, of the relation between the argument of a function and the value it determines to the relation of part and whole. For although the function 'capital of . . .' takes the value Stockholm when Sweden is taken as argument, Sweden is not part of Stockholm. Dummett's discussion of this issue has permanently disposed of the temptation to take these remarks seriously as interpretive constraints. We shall see below, however, that there remain genuine controversies which are not so easily disposed of (concerning the senses and referents of functional expressions) which may be regarded as successors to these two mistaken lines of thought.

Dummett has shown that Frege should be treated as a modern thinker in the sense that one can think about contemporary philosophical issues of considerable significance by thinking about his concepts and their explanatory deployment, and that one cannot think about those concepts and their principles of deployment without thinking about such contemporary issues. In what follows, those concepts are approached from three different directions. First, an attempt to interpret and develop Frege's technical scheme in light of contemporary discussions of the issues he was addressing is considered. Then attention is turned to an argument to the effect that ignoring the historical context in which Frege developed his theories, treating him we might say *merely* as a contemporary, leads to substantive misinterpretation of those theories. Finally, following one strand of the account of the path by which Frege developed and defended some of his central concepts leads to a novel diagnosis of the status of those concepts.

I. Bell on Sense and Reference

One important offering is David Bell's book *Frege's Theory of Judgement*.[7] This is a clear and well-written work. The issues it raises and the form in which they are addressed merit the attention of anyone interested in the significance for current inquiry of Frege's strategic deployment of a battery of technical concepts to explain various aspects of linguistic practice. Its title is worthy of some consideration. It is a measure of the degree of sophistication of contemporary Frege commentary that a controversy exists even over how one should describe the topic which his philosophical work addresses. Of course no one disputes his concern with the foundations of mathematical reasoning and knowledge, expressed above all in his three books, the *Begriffsschrift*, the *Grundlagen der Arithmetik*, and the *Grundgesetze der Arithmetik*. But the more general conceptual framework he found it necessary to elaborate in order to express clearly and precisely his claims about the nature of mathematics and its objects cannot easily be characterized without prejudging substantial issues of interpretation. It may seem obvious that Frege was pursuing a project in the philosophy of language.[8] But such a description is misleading in the context of Frege's own insistence on the priority of thoughts (though not of thinkings) to their linguistic expression. For he was interested in natural languages only insofar as they permitted rough formulation of objective and language-independent thoughts, and he crafted artificial languages only as more adequate means for their expression. It would be inappropriate to build into the description of the subject matter at the outset a post-Wittgensteinian conviction of the wrongheadedness of such an approach by assimilating his concerns to contemporary investigations under the rubric "philosophy of language." One of the major theses of Hans Sluga's book, discussed below, is that such Whiggish presuppositions of continuity of concern have consistently led Frege's readers to overlook important strands of his thought. Dummett has also suggested "theory of meaning" as a general characterization, but this seems to apply better to his own enterprise than to Frege's. For 'meaning' is correlative to 'understanding', and Frege's concern lay at least equally with reference, which is not in general grasped when one understands a claim, as with the sense which must be grasped in that case.

In his discussion of the book,[9] Dummett objects that Bell has misdescribed his topic, in that Frege's treatment of the act of asserting is the topic of only one chapter, while the rest of the book talks about the notions of sense and reference. This seems unfair, for the heading "theory of judgment" ought to entitle Bell to offer an account of the contents which are judg*ed* as well as of the acts which are the judg*ings* of those contents. It has the advantage of placing Frege's concerns in appropriate historical and philosophical context. Bell's denomination of Frege's topic as judgment displays his recognition of the importance Frege, in company with Kant and Wittgenstein, placed on inverting the traditional order of explanation which took concepts as primary and sought to account for judgments in terms of them. At least until 1891, Frege clearly

regarded the claim that concepts can only be understood as the products of analysis of judgments as one of his most central insights. Although Bell does not say so, it is equally clear in the *Begriffsschrift (BGS)* that Frege completes the inversion of the classical priority of concepts to judgments and judgments to syllogisms by taking the contents of sentences (judgment in the sense of what is judged rather than the judging of it) to be defined in terms of the inferences they are involved in.[10] Concepts are to be abstracted from such judgments by considering invariance of inferential role (which pertains only to judgments) under various substitutions for discriminable (possibly nonjudgmental) components of the judgment. Both in the introduction to *BGS* and in his essay "Boole's Logical Calculus and the *BGS*,"[11] the virtue of the purely formal perspicuous language of inference in nonformal contexts is described as its permitting for the first time the scientific formation and expression of concepts. Although it is for this reason that Frege called his first work a "concept script," he later came to believe this phrase misleading precisely because it obscured his doctrine of the primacy of judgments. It would be equally misleading, however, to describe Frege simply as a theorist of inference, in spite of the explanatory priority he accorded to it. For his primary theoretical focus always lay on the sentential and thence subsentential contents attributable to different expressions in virtue of the roles they played in inference, as revealed by their behavior under substitution. So "judgment," which is (a translation of) an expression Frege himself used pretheoretically to describe the object of his theorizing, seems a good choice to delimit his subject matter.

Like any other choice, however, it does prejudge some controversial issues of interpretation, for instance, that concerning the persistence in Frege's thought of the so-called "context principle." It is often unclear exactly what this principle means, but the canonical statement of it is the *Grundlagen* claim that "only in the context of a sentence does a word have any significance." (I use 'significance' here for Frege's '*Bedeutung*' because in 1884 he had not yet distinguished *Sinn* from *Bedeutung*, and the undifferentiated term should be marked.) It is often claimed,[12] even by those such as Dummett who take the putative change in view to be a serious mistake, that when Frege achieved his mature views in 1891 with the formulation of that crucial distinction he discarded the context principle. If that is so, then Bell's choice of "theory of judgment" to describe the topic of the mature semantic views he discusses would be misleading or simply incorrect. As we shall see below, Sluga argues that Frege never relinquishes the context principle. Bell does not argue this, however, nor does he even claim it. He is simply silent on this issue, as on others concerning detailed questions about the attribution of various views to Frege based on textual evidence.

Bell's enterprise lies in a different direction entirely. He is concerned to look closely at the explanatory roles played by Frege's various concepts and at the ways in which Frege takes them to be related, in order to refine and reconstruct a broadly Fregean account of the nature of judgment. In keeping with this aim, he is not engaged in the exegesis of Fregean texts, and freely discards from his reconstruction a number of doctrines which Frege clearly held, in favor of incompatible principles (for instance, in Bell's reconstruction functional expressions are assigned senses but not referents). His project is to salvage from Frege's account those insights which can be put together to form a workable theory of judgment. The result is broadly Fregean in endorsing the following "major strands" of Frege's theory:

1. There is the methodological principle that 'we can distinguish parts in the thought corresponding to the parts of a sentence, so that the structure of the sentence serves as a model of the structure of the thought'.
2. A thought is (a) objective, (b) the sense of an indicative sentence . . .
3. A thought must have at least one 'unsaturated' or functional element, otherwise its elements would fail to coalesce and would remain merely disparate atoms.
4. In a thought the complete elements refer (if at all) to objects.[13]

The nature of this enterprise makes it hard to evaluate its success. There are many issues one would think to be central to any attempt to offer a theory of judgment which Bell nevertheless does not address. For instance, although he argues that it would be wrong to require an account of judgment to restrict itself to the form of an account of the propositional attitude constructions used to attribute judgments to others, he does not justify the book's failure to present any such account as a proper part of such a theory. Again, although it has been suggested above that Bell was not obliged to restrict his attention to the notion of

assertoric force (the analysis of the act of judging), one would certainly like a fuller and more satisfactory account of that notion than the cursory sketch we are offered.[14] The book does its work in a sort of methodological no-man's land between textual exegesis and theory construction owing allegiance only to the phenomena it seeks to theorize about.

This is not to say that the analysis is not enlightening, however. Bell is at his best when dissecting the explanatory role assigned by Frege to his technical concepts. When he succeeds, we learn both about Frege and about the phenomena. Consider for instance the notion of *Bedeutung*. Bell tells us that

> Frege had not one, but two notions of reference. These notions hang together so well in the case of singular terms that they are hard to distinguish in this context. In the case of predicates, however, they are not only distinguishable, they are difficult to reconcile. One notion is this: the reference of an expression is that extra-linguistic entity with which the expression has been correlated or which it picks out. The other notion of reference is that it is a property which an expression must possess if that expression is to be *truth-valuable* (to coin a phrase). By truth-valuable I mean such that it either possesses a truth-value, or is capable of being used (and not just mentioned) in a sentence which possesses a truth-value.[15]

Bell claims that although in the case of singular terms one notion can play both of these roles, since for them to be truth-valuable just *is* to be correlated with an object, in the case of sentences and functions the two notions diverge. All that Frege ever offers in the way of evidence for the application of the notion of reference to expressions in these categories is considerations showing them to be truth-valuable. Since he does not distinguish the two different notions of reference which he has in play, he feels entitled to conclude that they possess reference in the first sense as well. But this is a non sequitur, or at any rate a transition which must be justified and not simply assumed on the basis of the conflation of the two different senses of *Bedeutung*. Thus Bell rejects the notion of truth values as objects, and of functions as the references of functional expressions, as excess conceptual baggage mistakenly mixed in with the second notion of reference, which is the only one doing any explanatory work for these categories.

This analysis is clearheaded and valuable but can be faulted on two grounds, each of which amounts to a request for further analysis. First, as Dummett points out,[16] the characterization of the second notion of reference does not seem right. For as Bell has described it, reference is a property which an expression either has or lacks, depending on whether sentences containing it can have or always lack truth values. But Frege's notion is that in addition to having or lacking reference, expressions which have reference can have *different* references, accordingly as they make different contributions to the truth values of sentences containing them. The test is always substitutional: two expressions which have reference have different references if and only if in some context the substitution of one for the other changes a true sentence into one which is not true. Others who have noticed the distinction Bell is after have put things better. For instance, Ernst Tugendhat[17] (who seems to have introduced this line of thought) calls this nonrelational sense of reference "truth-value potential" and in effect identifies the truth value potential of a subsentential expression with the equivalence class of expressions intersubstitutable *salva veritate.*

The sharpening of Bell's distinction (which makes it similar to that between 'referent' and 'reference' which Dummett uses throughout *Frege: Philosophy of Language* [see note 2]) does not affect his criticism of the inference from possession of reference in this nonrelational sense to possession of reference in the relational sense, of course. But it does affect a further use he wants to make of the distinction to argue that it is incorrect to think of predicate expressions as having a reference at all, even in the nonrelational sense. For here Bell argues that Frege incorrectly takes as a necessary and sufficient condition for the truth-valuability (in Bell's sense) of predicates that they have sharp boundaries. He accordingly takes it that the assignment of reference to predicates is motivated only by this requirement, and so showing the untenability of such a requirement is sufficient to show the inappropriateness of assigning reference to predicate expression at all. This line of argument is undercut by seeing that there is more to the second notion of reference than truth-valuability. Since the denial of the cogency of the application of the notion of reference to predicates (or function expressions generally) is one of the main innovations of Bell's analysis, his failure adequately to characterize that part of Frege's notion of reference which remains when one takes away correlation with an extralinguistic object has serious consequences for the subsequent course of his argument.

Dummett, however, rejects not only Bell's characterization of the second notion of reference but also the claim that there are *two* notions of reference. He claims that the relational and the nonrelational senses represent "two ingredients of *one* notion." The second "tells us what Frege wanted the notion of reference *for,* and the other tells us how he thought that it applied to the various categories of expression."[18] It may be granted that the explanatory work Frege wanted the notion of reference for is its truth value potential or contribution to the truth conditions of sentences, and that he thought that the intersubstitutability equivalence class of equipollent expressions was determined by the correlation of all and only its members with the *same* extralinguistic entity. But it would still remain to be asked, for instance, whether the identity of the correlated object and the nature of the correlation can be inferred from the semantic equivalence class of expressions they determine, as Frege's arguments concerning the reference of sentences and functional expressions would seem to require. Such a question is in no way made less urgent or easier to answer by rephrasing it in terms of two ingredients of one notion rather than in terms of the relations of two notions. In the final section of this chapter I will argue that this difficulty is one instance of a quite general definitional failure of Frege's part, one which in another context he tried unsuccessfully to resolve in a purely technical way.

Putting the issue in these terms raises the second source of dissatisfaction with Bell's argument. For the sort of question just raised seems no less important or difficult for the paradigmatic case of singular terms than for the parts of speech Bell finds problematic. The basic substitutional/inferential methodology which yields the nonrelational sense of reference as an equivalence class of expressions vastly underdetermines the correlated objects and mode of correlation invoked by the relational sense even for proper names. Tugendhat, having formulated the nonrelational notion of reference, takes it to be *the* notion of reference, discarding correlation with an object as a realistic confusion best extruded from Frege's thought. Sluga follows Tugendhat in this regard. The reason in each case is that all that Frege's analysis of the use of expressions seems to require is the sorting of expressions according to the nonrelational sense of substitutional role. The semantic analysis he developed is a method for the perspicuous codification of inferences. Truth is what is preserved by good inferences, and subsentential expressions can be grouped into co-reference classes accordingly as intersubstitution within the classes preserves such good-inference potentials. Such an approach can give rise to specification of the conditions under which two expressions have the *same* reference, but how can it warrant a claim that the shared reference is to be identified with some object (among all those which in one way or another could be taken to determine the same co-reference classes) specified otherwise than as the reference of an expression? The answer seems to be that Frege's arguments for this identification are straightforwardly substitutional ones, in particular, that for any singular term *t* we can always substitute (saving the inferential potentials) the term *the object referred to by the singular term 't'*. The expressions which license intersubstitution of expressions are identity locutions (as Frege had argued in the *Grundlagen*), and so we are correct to say that the object referred to by the singular term 'Julius Caesar' *is* Julius Caesar. Whether this fact has the significance Frege thought it had is another matter.[19]

One of the most important discoveries of the early 1970s, from the point of view both of the interpretation of Frege and of the philosophy of language generally (for once, made independently of Dummett), concerns the need to distinguish two different explanatory roles which are conflated in Frege's technical concepts of sense. Saul Kripke and Hilary Putnam independently argued[20] that the *cognitive* notion of the sense of an expression, what one who has mastered the use of that expression may thereby be taken to understand and the *semantic* notion of the sense of that expression, what determines the reference of the expression, cannot in general be taken to coincide. In particular, in the case of proper names, no knowledge or practical capacity which can plausibly be attributed to an ordinary competent user of the name will suffice to determine the object of which it is a proper name. A similar point can be made about the use of natural kind sortals. Since Frege had required that his notion of the sense of an expression play both the cognitive and the semantic role, and since for an essential range of expressions no single notion can do so, it is apparent that his concept must be refined by dividing it into two distinct sense-concepts, whose interrelations it then becomes urgent to investigate.

A further distinction within the semantic notion of sense has been urged by a number of writers on the basis of the consideration of the behavior of indexical or token-reflexive expressions.[21] In Kaplan's idiom,

we must distinguish for such expressions between their *character*, which is associated with the expression type, and the *content* associated with each contextually situated token(ing) of that type. The distinction in question is evident in the following dialogue:

A: I am anxious to get started.
B: No, it is possible that you are eager, but I am the anxious one.

We are concerned with the semantic notion of the sense of an expression, that is, with the way in which its reference is determined. In one sense both tokens of "I" have their reference determined in the same way, for in each case it is the speaker responsible for the tokening who is referred to. These expressions share a character. But in another sense A's token of "I" and B's token of "you" have their reference determined in different ways (e.g., for the purpose of tracking the referent through the other possible worlds which must be considered to evaluate the modal qualifications in B's remark). The referents of these tokenings will coincide in every possible world relevant to the evaluation of these utterances, in virtue of the identity of their contents. The characters of these expressions, together with the context in which they are uttered, determine a content which in turn determines a referent in every possible world. It is this latter task with which the semantic notion of sense is charged for nonindexical expressions. Such expressions may accordingly be thought of as those whose character determines a content without needing to be supplemented by a context. The point is that as we ask about what would be true in other worlds of the individual picked out by B's indexical utterance, there is a *double* relativity to possible worlds, accordingly as those worlds can be relevant to the two different stages in the determination of a referent. First, since B's remark could have been addressed to someone other than A, we must consult the world-context in order to determine what content is fixed by the character of the expression when uttered in that context. The individual concept so determined as a content can then be tracked through various possible worlds and assigned referents in each, so that modal claims can be evaluated.

Without referring to either of these antecedents, Bell distinguishes two notions of expression sense in a way which partakes of some of the features of each of the other distinctions. He calls his two notions "input sense" and "output sense," and introduces them by reference to two Fregean principles:

PS1: The sense of a sentence is determined by the senses of its component parts,

and

PR1: The truth-value of a sentence is determined by its sense. (And, of course, how things stand.)[22]

His claim is that although the "two principles depend for their plausibility and usefulness on there being a sense of 'sense' which remains constant throughout," in fact they demand different ones. Input sense is that notion of which principle PS1 holds, and output sense is that notion of sense of which PR1 holds. Input sense is that which is preserved by correct translations and that for which synonymy claims assert identities of sense. Subsentential expressions have input senses ("meanings"), and these combine to determine the input senses of sentences containing them. Output senses are defined as what is common to claims such as "Today I ate plum pudding," and "Yesterday you ate plum pudding." The input senses of sentences together with a context of utterance determine such output senses. The output senses of sentences are what can meaningfully be described as true or false, as per principle PR2.

As described so far, Bell's distinction amounts to the claim that the cognitive/semantic and character/content partitions of the notion of sense ought to be seen as coinciding. For the compositionality of 'sense' is a postulate required for the explanation of the possibility of understanding complex expressions, so that it must be input senses which are in the first instance grasped cognitively. Semantic senses, determining truth values of sentences, are in turn identified with output senses. But since the latter are determined by the former together with a context of utterance and the distinction is enforced by attention to indexical expressions, the character/content distinction is likewise subsumed by the difference between input and output senses.

Such an identification is clearly subject to a number of objections, as consideration of the quite different motives and functions of the conflated distinctions indicates. But these difficulties may not be insurmountable. Perhaps a useful view could be elaborated based on the assimilation of the sense in which the referent of a proper name token is determined not by what its utterer understands by it, but only by this together with a causal, historical, and social context in which the token is embedded, on the one hand, and the sense in which the reference-determining sense of a token of "yesterday" is given not just by what one can understand as the meaning associated with the expression type, but only by this together with a concrete context of use. But Bell does not attempt to develop such an account. In part this is because he has nothing whatever to say about what "contexts" are, or how these together with input senses determine output senses. And it is just here that all the detailed work is involved in making out either half of such an assimilation, and hence in justifying their conflation. But Bell is precluded from addressing such a task by other, less defensible features of his view.

For Bell denies that subsentential expressions have output senses at all, claiming that "output sense is essentially sentential."[23] No argument or even motivation for this position is presented. It is suggested that for sentences the distinction between input senses and output senses corresponds to that between sentences and the statements they can be used to make, and that it is better to think of the former not as possessing truth values which change, by contrast to statements whose truth values do not, but rather to think of the former as not the kind of thing which can have truth values at all. But no reason is given for not extending this distinction to subsentential expressions. The distinction between the two varieties of sense is introduced, as indicated above, in terms of two Fregean principles. PR2, the 'sense determines reference' principle, is quoted at this portion of the argument as restricted to sentences and truth values. But of course the principle Frege uses is not so restricted. Indeed, when Bell first introduces it some sixty pages earlier, it is in unrestricted form. He has just been discussing the principle he calls PR1, that the reference of complex expressions is determined by the references of their components (which Bell discards because as we have seen he does not attribute reference of any kind to functions). He says:

> Elsewhere in his writings, however, he seems to invoke a quite different principle which we can call PR2. It is this: (a) the reference of any expression is determined by its sense, (b) the sense of a complex expression is determined by the senses of its component parts.[24]

Two features of *this* definition deserve comment. First, part (b) of principle PR2 as here stated is what he later calls PS1 and is concerned precisely to distinguish from PR2. Second, part (a) of *this* original statement differs from the later version in not being restricted to sentences.

Neither of these substantial changes in the significance of his expression "PR2" is announced, acknowledged, or motivated in the intervening text. Such carelessness in specifying a central interpretive principle which one has taken the trouble to name for clarity of reference is bad enough under any circumstances. It is unforgivable when essential features of one's own claims and their justifications depend precisely on the matters obscured by the sloppiness. As things stand, the reader is left with no idea why in using the two principles PR2 and PS1 (= PR2(b) in the earlier statement) to distinguish two notions of sense one should employ the later version of PR2 rather than PR2(a) from the earlier version, which is the principle Frege endorsed. Apart from the invocation of PR2, output senses are specified as what is common to the two "plum pudding" sentences quoted above. As my sketch of the character/content distinction shows, it is not at all obvious why this characterization should not extend to what is common to 'today' and 'yesterday', on the one hand, and 'I' and 'you', on the other.

Bell does, however, employ the restriction of output senses to sentences to argue for a further point. For he claims that the "context principle" of the *Grundlagen* may be understood in terms of the fact that terms only have input senses, which together with the input senses of other expressions determine sentential input senses, which in context determine a truth value. Since the reference of terms matters only in determining truth values, it is "only in the context of a sentence that a term have a reference." Clearly nothing can be made of this line of thought in the absence of a rationale for its basic premises.

These difficulties with the distinction between input senses and output senses also make it difficult to evaluate another novel interpretive suggestion which Bell offers. He concludes his discussion of the senses of proper names with the claim "The sense of a proper name, then, is that it purports to refer to a determinate object of a given sort with which it has been conventionally correlated."[25] The sense of a proper name is here taken as "that which one understands when one is able to use it correctly."[26] As indicated above in the discussion of the relation of the cognitive notion of sense to Bell's notions, this must be the input sense, for subsentential expressions are not supposed to have output senses. It is accordingly obscure what the connection is supposed to be between the senses Bell is offering a theory of here and the determination of referents for the proper names they are senses of. What then are the criteria of adequacy for an account of what a name user must be taken to understand? Bell examines the conditions under which we would want to deny that someone had mastered the use of a name, and concludes that in addition to using it as a singular term, one must at least know some sortal under which the referent is taken to fall in order to be judged a competent user. This is useful as a necessary condition, but much less plausible as a sufficient condition to be taken to be using an expression as a proper name. For a sufficient condition would seem to require that one be appropriately connected to a community of users of the name, perhaps a historically extended one, whose joint use *does* determine a referent, though no individual's use need do so. It is not obvious that merely believing that some conventional correlation has been established with an object of the right sort is sufficient to be appropriately connected with the community of users of that name. In any case, to argue for such a principle would require looking at how input senses and various specific sorts of context can together determine output senses and eventually referents for the names in question, and this Bell does not undertake.

Bell wants his notion of proper name sense in order to develop an appropriate account of the senses of functional expressions. This latter task is made especially urgent by the confrontation between his denial that the referents of functions have any explanatory value, on the one hand, with the undeniable importance in Frege's scheme of functions and concepts understood as functions, on the other. Bell's reconstruction reconciles these ideas by interpreting concepts and functions as the *senses* rather than the references of functional expressions. A concept, accordingly, is to be understood as a function which can take as arguments proper name senses of the sort he has described, and yield thoughts, the senses of sentences. While this identification of concepts must be seen as a revision rather than an interpretation of Frege's thought, it might seem that, setting that identification aside, at least the account of the senses of functional expressions as functions from the senses of argument expressions to the senses of value expressions ought to be uncontroversial. It is not, and it is instructive to see why not.

As Bell has pointed out in his discussion of senses generally, the concept of sense is required to play two distinguishable roles. First, the sense of a component of a complex expression must contribute to the determination of the sense of that complex. But also, the sense of the component must determine a reference for that component. This gives us two different ways to think about the senses of functional expressions such as predicates. On the one hand, they must combine with the senses of terms to yield the senses of sentences. On the other hand, they must be the way in which a function from objects to truth values is determined or given. It is not obvious that these two jobs can be done by one notion. In particular, Dummett has argued that "once the proper name has specified the way in which the object is given, then it has made its contribution to the sense of the sentence; if it had not, then it would be impossible to see how its sense could *both* contribute to the sense of the sentence *and* consist in the way in which the object is given."[27] That is, maintaining the coincidence of the two roles of sense in the case of proper names (presumably where our grasp is firmest) commits us not only to their divergence for functional expressions, but also to which half we give up, namely, the identification of their senses with sense functions. Peter Geach has objected to this doctrine of Dummett's,[28] and it is instructive to examine Dummett's response.

It is not disputed that once a sense has been assigned to a predicate, a function from the senses of proper names to thoughts is determined. For according to Dummett, the predicate sense is the way in which a function from objects to truth values is given. Hence, when that function is supplemented by an object, it determines a way in which a truth value is given, that is, a thought. But since a term sense will determine such a supplementing object (according to the second role of senses mentioned above), the predicate sense will induce indirectly a function from term senses to sentence senses. As Dummett says, "The question is whether the sense of the predicate just *is* that function."[29]

To argue that it is not, Dummett appeals to a further thesis of Frege's about senses, namely, that the senses of component expressions are *parts* of the senses of the complex expressions in which they occur. We have seen that it is a mistake to think of functions or their arguments as parts of the values they generate, as Frege's retraction of his careless claim that objects are parts of truth values shows. But since Frege did hold that predicate senses are parts of thoughts, we would be committing precisely this howler if we identified those senses with functions taking term senses into thoughts. This is an ingenious counterargument, but it cannot be considered decisive. For while it would be a howler to treat functions and their arguments generally as parts of the values they determine (as in the combination of Sweden and the function *the capital of*... to yield Stockholm), this consideration does not show that particular functions and kinds of function cannot have values which contain the functions or their arguments as parts. Stockholm is part of the value of the function *the country of which*... *is the capital*. And mathematical examples of function-values which contain functions as parts in the set-theoretic sense are easy to come by. (One thinks of the story of the oracle who offered to answer a single question, and upon being asked "What is the ordered pair whose first element is the best question I could ask you, and whose second element is its answer?" replied—falsely, I suppose—"The ordered pair whose first element is your question and whose second element is this answer.")

Insulated from this dispute about sense functions by his distinction between input senses and output senses, Bell backs up his commitment to treating the senses of functional expressions as functions by citing a number of passages, both published and from the posthumous works, in which Frege unequivocally describes such senses as "unsaturated," "incomplete," and "in need of supplementation," going so far in fact as to say that "the words 'unsaturated' and 'predicative' seem more suited to the sense than to the reference."[30] To motivate his identification of concepts with sense functions, Bell argues as follows.[31] The only reason Frege had for believing in concepts as predicate referents was the need to deal with a situation in which predicates have a sense and so determine a thought, but lack a reference, and so determine a thought which has no truth value. The only case where this can happen which does not reduce to the failure of a term to have a reference is where the predicate is not defined for the sort of argument to which it is applied. But this sort of case can be much more plausibly excluded by considerations concerning predicate *senses*. For such cross-categorial predications (such as "Julius Caesar is the sum of two prime numbers") ought properly to be seen as not succeeding in expressing thoughts at all. Bell's solution accordingly is to see predicates as having sortal restrictions associated with their argument places, which together with the 'sortal physiognomy' he has already assigned to proper name senses yields the result he desires. One of the benefits which might be derived from such a radical reconstruction should be made manifest by the discussion to be given below of the difficulties ensuing from Frege's insistence that functions be defined for all arguments whatsoever. As before, however, the evaluation of this thesis about senses must await some resolution of the general questions Bell has left open concerning his distinction between input and output senses.

II. Sluga on the Development of Frege's Thought

Hans Sluga's book on Frege in the "Arguments of the Philosophers" series[32] represents an approach complementary to Bell's in almost every regard. Its central aim is to reread Frege's work in the light of that of his precursors and contemporaries, rather than by reference to his successors in the analytic tradition, as has been traditional. Although Frege's unprecedented innovations in symbolic logic have made it natural to think of him exclusively in the role of the founder of a tradition—as a man without a past—Sluga argues that we ignore at our peril his intellectual climate and the influences which conditioned various aspects of his technical concepts and of the explanatory tasks he set for them. Sluga's task is not purely historical, however. For he is also concerned to set out and justify novel readings of some of Frege's purely philosophical doctrines, readings which are suggested and motivated by the historical recontextualization he recommends. The result is a stimulating new picture of Frege's thought which will be of interest even to those who are not in the end persuaded in detail by it. Furthermore, since the narrative strategy employed is to trace the development of Frege's ideas chronologically (starting, as it were, before he was born) and surveying all of his important writings *seriatim*, this book is excellently constructed to serve as an introduction to these ideas (as Bell's or Dummett's books, for instance, could not) as well as to challenge specialists.

The book's historical orientation, then, is adopted not only for its own sake, but also in order to guard against blinding ourselves to interpretively significant features of Frege's work by the importation of anachronistic prejudices. Accordingly, it is primarily in terms of the philosophical illumination they provide for our appreciation of Frege's concepts and claims that we must evaluate the success of Sluga's various invocations of historical influence. The claimed influences may be considered under four headings. First, a view is presented about who Frege took to be his philosophical opponents. Next, Leibniz is identified as a precursor. Third, claims are made about the influence of two logicians of the generation preceding Frege's, Lotze and Trendelenburg. Finally and

most significantly, it is claimed that overlooking the intellectual debt which Frege owes to Kant has most seriously distorted our understanding. I will consider these claims in this order.

In his first chapter, Sluga is concerned to refute the claim that "in a history of philosophy Frege would have to be classified as a member of the realist revolt against Hegelian idealism, a revolt which occurred some three decades earlier in Germany than in Britain."[33] In this aim he succeeds unequivocally. Hegelianism had ceased to be dominant or even popular in German philosophical circles some years before Frege was born. The view against which Frege was reacting is the scientific naturalism which Sluga claims was held by the physiologists-turned-philosophers Vogt, Moleschott, Buchner, and Czolbe, popularized during Frege's lifetime by Haeckel, and shared with some reservations by Gruppe. Ontologically this view is a reductive materialism, and epistemologically it is an empiricist psychologism. Sensations are viewed as material processes of the brain. Concepts, and hence the thoughts constructed from them, are taken to be reflections of such sensations. Logic is seen as the study of the laws of thought, that is, as an empirical investigation seeking to establish the natural laws governing the association of concepts in judgment and of judgments in inference. It is this psychologism which Frege so vigorously opposed, and on those relatively few occasions when he describes his opponents as 'idealists' it is clearly this school which he has in mind.

This is a point of no small moment, especially in the context of an evaluation of Frege's role as progenitor of the analytic tradition. For his overarching objection to the naturalists is their failure appropriately to distinguish between the normative and ideal order of correct inference and justification on the one hand, and the descriptive and actual order of causation and empirical processes on the other. Their concommitant confusion of features of cognitive acts with features of the contents of those acts is merely the expression of this original sin. And in his insistence on the centrality of this basic distinction, Frege is at one with Kant and the post-Kantian idealists, and at odds with the primarily physicalist and empiricist tradition in Anglo-American philosophy which he fathered, and in the context of which it has been natural for us to read him.[34]

Throughout his book Sluga talks about Leibniz's influence on Frege, but when he specifies the details of this influence, his claims turn out to be quite weak. Like Leibniz (and Kant), "Frege is interested in the study of logic and the foundations of mathematics because they allow one to ask in a precise form what can be known through reason alone."[35] Aside from this general rationalist commitment to the possibility of a priori formal knowledge, the only Leibnizian doctrine which is attributed to Frege is the endorsement of the project of the universal characteristic. Frege explicitly describes the motivation for his *Begriffsschrift* in this way. That at this level of generality Frege owes a debt to Leibniz is hardly a novel or surprising claim, however. Sluga also discusses the influence of Trendelenburg, but in the end the claims seem to come to little more than that he was the conduit through which Frege became familiar with Leibniz's ideas.

It is otherwise with the connection discerned between Frege and the logician Hermann Lotze. The suggestion of influence here has specifically been denied as "a remarkable piece of misapplied history."[36] Yet in this case Sluga shows sufficiently striking similiarities to make the hypothesis of influence persuasive. It is known that Frege read Lotze. Indeed it has been argued that the theory of judgment in opposition to which he presents his innovation in the *Begriffsschrift* just is Lotze's formulation.[37] The essay immediately preceding "The Thought" in the journal in which it was originally published, which Sluga takes to have been intended by the editors as an introduction to Frege's essay, mentions Frege in the context of an exposition of Lotze which highlights several Fregean doctrines.[38] From Sluga's account of Lotze's views (as presented in the *Logik* of 1874 and an earlier work of 1843), one can extract eight points of similarity with Frege.

First, Lotze inveighs against psychologism and indeed is the figure Frege's contemporaries would probably have identified as leading the battle against the dominant naturalism of the day and in favor of a more Kantian position. Second, Lotze was a logicist about mathematics, although there is no hint in his works that he took the detailed working out of such a reduction to logic as part of what would be required to justify this view. Third, Lotze insists, against empiricistic sensationalism, on the distinction between the objects of our knowledge and our recognition of such objects, in much the same terms that Frege did. Fourth, Lotze emphasized and developed the Kantian strategy of explaining concepts as functions (though of course he does not have the notion of functions as unsaturated which Frege derived from his own substitutional method of assigning contents to subsentential expressions). Fifth, Lotze attacks the empiricists with a distinction between the causal conditions of the acquisition of concepts and the capacity to use such concepts in correct reasoning which mastery of the concepts consists in (see note 34). Next, Lotze offers a theory of identity statements according to which the two terms share a content, but differ in form. This is the *Begriffsschrift* view, and the language survives into the opening paragraphs of "Über Sinn und Bedeutung." Seventh, Lotze endorses the Kantian principle of the priority in the order of explanation of judgments to concepts which Frege endorses in the *Grundlagen*. Lotze does not succeed in being entirely consistent on this point, since he also is committed to atomistic principles which are not obviously compatible with the view on the priority of judgments. Although Sluga does not say so, those who take Frege not to have discarded the context principle in the post-1890 writings must find a similar tension in some of the procedures of the *Grundgesetze*. Finally, Lotze is committed to the objectivity of sentential contents, and treats them as neither mental nor physical just as Frege does. Lotze, however, specifically denies that this objectivity is grounded in the correlation of sentences with objects such as Frege's thoughts appear to be, taking a more Kantian position. Sluga, as we shall see below, argues that despite apparent statements to the contrary, we should understand this to be Frege's view as well.

This is a suggestive set of similarities to find in a prominent near contemporary logician with whose work Frege was familiar. Recognizing them as important need not commit one to minimizing the significant, perhaps dominant, differences in outlook which remain between Lotze's revived Kantianism and Frege's philosophical elaboration of his semantic methodology (although Sluga does on occasion succumb to the temptation to treat Frege's agreement with Lotze on one point as evidence that he probably agreed with him on others). Only according to the crudest notion of what philosophical originality consists in is there any incompatibility between finding enlightenment in the demonstration that these general principles were in the air and so came complete with a history and a tradition on the one hand, and the appreciation of the genius shown in the use such adopted and adapted raw materials were put to in the service of quite a different explanatory project on the other.

Sluga's most important and sustained argument, however, concerns the influence of Kant on Frege. He claims that Frege should not be thought of as a dogmatic realist about physical objects nor as a Platonist about abstract objects, as he almost universally has been thought of. He should be seen rather as a Kantian whose realistic remarks are to be interpreted as expressing that merely empirical realism which is one feature of transcendental idealism. This is certainly a radical reinterpretation. What evidence can be adduced for it? Sluga's considerations may be assembled as five distinct arguments.

First, it is pointed out that Frege joined a philosophical society whose manifesto is explicitly idealist and Kantian, and that he published in its journal. By itself, this shows little, for Frege had so much trouble getting his work into print and finding others willing to discuss it that we cannot be sure how much he would have put up with to secure such opportunities. The rationale Sluga suggests[39] is that "what tied him to the idealists was primarily his opposition to the various forms of naturalism." Specifically, Frege and the idealists (a) were anti-psychologistic, (b) endorsed an objectivist epistemology (taking the contents of judgments to be independent of their entertainment by thinkers), and (c) endorsed a rationalistic a priorism about mathematics. These points are well taken, but the views involved are all consistent with Platonism and realism generally as well as with transcendental idealism. Indeed Sluga admits that "one can read much of Frege and not raise the question of transcendentalism." So we must look elsewhere for a warrant for such an attribution.

The second argument concerns Frege's attitude toward the truths of geometry.[40] It is remarked to begin with that in his *Habilitationsschrift* Frege held a Kantian view on this topic, saying that geometry rests "on axioms that derive their validity from the nature of our capacity for intuition *(Anschauungsvermögen)*." Furthermore, throughout his career Frege describes geometrical knowledge as synthetic a priori, and on this basis rejects non-Euclidean geometry as *false*. From this fact Sluga concludes: "Frege held a Kantian view of space and hence a transcendentally subjective view of the objects that occupy it." The only elucidation offered of this crucial "hence" is the later statement that "Frege's view must be close to Kant's: Empirical objects are in space and time, but space and time are a priori forms of sensibility. That seems to imply that for Frege empirical objects can only be empirically real, but must be transcendentally ideal." That Kant believed the two views to be linked in

this way falls far short of showing that Frege did so. Certainly such an argument cannot be taken to undermine an interpretation which takes Frege's realistic remarks about physical objects at face value, and admits that his views are inconsistent to the extent that he never confronted these latter with his views about geometry with an eye to reconciling them. Nevertheless, some interpretive cost is clearly associated with attributing such an inconsistency to Frege.

The next two arguments must be judged less satisfactory.[41] First, Sluga argues that in the context of Kantian transcendentalism (as just discussed), Platonic realism looks like dogmatic metaphysics. So Frege should have been expected to argue that views (a) through (c) above, on which he argues with the idealists, cannot in fact be warranted transcendentally. But Frege nowhere argues this. The trouble with this argument is that there is no evidence that Frege did not, as most of his contemporaries did, read Kant's transcendentalism as a form of psychologism. If he had done so, he would have dismissed it and so not felt the force of the demand in question. Sluga next argues that every claim of Frege's that can be taken as evidence of Frege's realism can be matched by a passage in Lotze, who had a Kantian idealistic theory of validity. This argument seems to do no more than restate the point that views (a) through (c) are consistent with either position. For it is a criterion of adequacy of anyone's transcendentally idealistic position that it have room for all of the claims the realist wants to make, suitably reinterpreted. Further, Frege does insist that thoughts are independent, not just of this thinker or that, but of the very existence or even possibility of thinkers at all. This seems to contradict Lotze's account of objectivity as rule-governed intersubjectivity.

Sluga's final argument is weightier and involves more interpretive work, in both construction and evaluation. The basic claim is that "there are strewn through Frege's writings statements that appear irreconcilable with Platonic realism. In particular the central role of the Fregean belief in the primacy of judgements over concepts would seem to be explicable only in the context of a Kantian point of view."[42] Arguing in this way obviously commits Sluga to showing that Frege does not discard the context principle when he arrives at the distinction between sense and reference. We will see below that he contributes significant new considerations to that debate in furtherance of this aim. But the incompatibility of realism with the recognition of the primacy of judgments must also be shown. The latter view is "Kantian," but it does not obviously entail transcendental idealism, which is the view in question. Sluga takes the principle of the primacy of judgments to serve the purpose for Kant[43] of refuting any atomistic attempt to construct concepts and judgments out of simple components, and in particular to resist the empiricist sensationalist atomism of Hume. Such a view is indeed incompatible with the reism of Tadeusz Kotarbinski (to which Alfred Tarski's recursive semantics owes so much), which sees the world as an arrangement of objects out of which concepts and judgments must be constructed set-theoretically.[44] But the Kantian principle need not be taken to be incompatible with Platonic realism about abstract entities such as *thoughts* which are the contents of judgments. Given that the context principle does not show that Frege was a transcendental idealist about thoughts, it seems also open to him to hold some form of realism about other objects, provided thoughts retain an appropriate primacy (as, given the very special status of truth in the late works, even those who see the context principle as discarded are committed to granting) even if he has not discarded that principle. So if the case for the persistence of the context principle can be made out, it should be taken as showing that. Frege was a Kantian in the sense of holding the context principle, not in the sense of being a transcendental idealist.

Still, this point is worth establishing for its own sake. Sluga correctly sees the *Begriffsschrift* as the confluence of three lines of thought: (1) that judgments, as involved in inference, are the original bearers of semantic significance, so that it is only by analyzing such judgments according to the procedure of "noting invariance under substitution" that such significance can be attributed to subsentential expressions ('the primacy of judgments'), (2) the Leibnizian idea of a perfect language, and (3) the idea of reducing mathematics to logic. Assuming the context principle was thus "anchored deeply in Frege's thought, it is implausible to conclude with Dummett that in his later years Frege simply let it slip from his mind."[45] Sluga advances five arguments for the persistence of the principle, and along the way addresses two commitments of Frege that have been taken to be incompatible with such persistence.

First, Sluga offers an important consideration which has not previously been put forward in the extensive literature discussing this question. The first of the 1891–92 essays that Frege wrote is a seldom read review of Ludwig Lange's *Historical Development of the Concept of Motion and Its Foreseeable End Result* titled "The Principle of Inertia." In it Frege argues at some length that the concepts of a theory are not given prior to and independent of that theory. Rather those concepts can be arrived at only by analyzing the contents which the judgments constituting the theory are given by the inferences concerning them which that theory endorses. This is a significant new piece of evidence supporting Sluga's view. The only question which might be raised about it is that since this semi-popular piece does not deploy the full-blown apparatus of sense and reference, it may be wondered whether the views there expressed were confronted by Frege with that apparatus, or whether the essay might not be seen as merely the latest of his early works. But to take such a line would be to concede a lot, and future claims that the context principle was discarded will have to confront this argument of Sluga's in detail.

Next Sluga offers a novel reading of the essay on the distinction between sense and reference which denies that, as has often been claimed, that distinction as there presented applies primarily to singular terms and their relations to the objects which are their referents, and hence commits Frege to an assimilation of sentences to terms which is incompatible with the context principle. The strategy here is, in effect, to deny that *'Bedeutung'* as Frege uses it *ever* has the relational sense which indicates correlation with an object. Relying on the Tugendhat essay mentioned above in connection with Bell, Sluga understands *'Bedeutung'* as a nonrelational semantic potential defined paradigmatically for sentences, in virtue of their role in inference. The introduction of this notion in the context of the consideration of identities involving singular terms is seen as a rhetorical device of presentational significance only. In the final theory subsentential expressions are taken to inherit indirect, inferential significances in virtue of their substitutional behavior in sentences, which alone are directly inferentially and hence semantically significant. Thus *'Bedeutung'* is paradigmatically a sentential notion.

To this analysis is conjoined an account of *'Sinn'* as a cognitive notion, as what matters for knowledge. But again, the units of knowledge are judgments, and subsentential expressions can become relevant only insofar as they can be put together to form sentences which can express judgments. So sense also should be seen as primarily a sentential notion, which applies to subsentential expressions only in a derivative way. This line of thought concerning senses is then combined with that concerning reference in a subtle and sensitive account of the puzzling relations between the Lotzean rendering of identity locutions offered in the *Begriffsschrift* and its successor in "Über Sinn und Bedeutung" ("ÜSB").

The previous discussion of Bell's interpretation suggests that these readings leave something to be desired. Sluga does not acknowledge the existence of any passage or considerations indicating that Frege does have a relational notion of reference in play. Yet such passages and considerations do exist, and merely elaborating the nonrelational version of Frege's concept, as Sluga does, does not obviate the necessity of investigating the relations between the two notions and the possibilities for reconciling them. Similarly, Sluga pushes his discussion of the notion of sense no farther than the discrimination of the cognitive role played by that concept. He has nothing to say about the semantic notion of sense, or accordingly about how senses are to be understood as determining references, even nonrelational references. On these points Sluga's analytic net does not have as fine a mesh as Bell's. As a result, his ingenious interpretation of sense and reference will require further filling in before its eventual promise can be assessed.

The overall interpretation which results from all of these arguments, however, is challenging and powerful. The primary objections to the persistence of the context principle are that Frege nowhere explicitly endorses that principle after the 1884 *Grundlagen* formulation, and that the principle is incompatible with two central doctrines of the 1891–92 essays, namely, the semantic assimilation of sentences to terms and the account of concepts as functions from objects to truth values. Sluga claims that his readings of the "Inertia" essay and of "ÜSB" meet these objections. He does not say in detail how the doctrine about functions is to be reconciled with the context principle, but does argue that the "Inertia" essay justifies us in attributing *Bedeutung* to any expression which makes an appropriate contribution to the possession of truth values by sentences containing it. Thus function-expressions may be assigned (nonrelational) reference on this account. Using intersubstitution equivalence classes to move from Tugendhat's nonrelational sentential semantic significances to those of subsentential expressions does indeed justify such an attribution. But in the "Inertia" essay, Frege seems to be using 'concept' in the ordinary sense rather than his technical one, that is, to refer to the *senses* of predicate expressions rather than their references. This being the case, it is not clear how the envisaged reconciliation of

the context principle with the view of concepts as functions from objects to truth values is to be achieved.

Besides the evidence of the essay on inertia, Sluga offers two further reasons to deny that the later Frege is silent on the topic of the context principle. First, he mentions in several places the posthumously published "Notes for Ludwig Darmstaedter" (of 1919) as showing that Frege continued to endorse the principle. He does not say what passages he has in mind, but he presumably intends the following: "What is distinctive about my conception of logic is that I begin by giving pride of place to the content of the word 'true,' and then immediately go on to introduce a thought as that to which the question 'Is it true?' is in principle applicable. So I do not begin with concepts and put them together to form a thought or judgement; I come by the parts of a thought by analyzing the thought."[46] Such a passage does show that sentences play a special explanatory role for the late Frege, but that much is not in question. At most such claims would show that a version of the context principle held for *senses*, confirming Sluga's claim that the cognitive origins of the concept of sense require that priority be given to sentences. No version of the context principle for referential significances follows from these claims. Unfortunately, Sluga never says what exactly he takes the context principle to be, whether a doctrine about senses, references, or both. Frege's original formulation, of course, preceded his making this distinction. So perhaps the best conclusion is that Sluga takes the principle to persist as applying to senses, that is, that it is only in the context of a thought that a term or other subsentential expression expresses a sense. This seems to be something Frege indeed did not surrender. Such a reading has the additional advantage that the doctrine that concepts are functions from the references of singular terms to truth values is not incompatible with it.

The final argument fares less well. It is claimed that Frege's late treatment of real numbers shows that his practice is still in accord with the context principle.[47] Here the point seems to be that the real numbers are given contextual definitions. Such an argument would be relevant to a context principle applying to reference rather than senses, since Frege does not pretend to specify the senses of numerical expressions in his formal definitions. But the definition of real numbers he offers is of just the same form as the *Grundgesetze* definition of natural numbers. If this style of definition does exhibit commitment to a form of the context principle, that case should be argued for the more central and important case of natural numbers. It is not clear how such an argument would go.

III. Frege's Argument

One of the themes around which Sluga usefully arranges his presentation of Frege's development is that of the pursuit of the definition of purely logical objects. The reason offered for the somewhat misleading order of presentation pursued in "ÜSB," which seems to give pride of place to singular terms rather than sentences, is that the road from the *Grundlagen* account of numbers to that of the *Grundgesetze* needed to pass through a more thorough understanding of identity claims. Sluga is quite clear that for Frege, beginning with the *Grundlagen,* the only concept we have of an *object* is as that which determines the semantic significance of a singular term. For an expression to play the semantic role of a singular term is for it to make a certain contribution to the inferential potential of sentences containing it, a contribution which is constituted by the appropriate (truth-preserving) substitutions which can be made for that expression. The substitution inference potential of a singular term is in turn codified in the endorsed identity claims involving that term. That what we mean by 'object' is according to Frege exhausted by our conception of that the recognition of which is expressed in identity claims in virtue of their licensing of intersubstitution is one genuinely transcendental element in his thought about which Dummett, Sluga, and Bell agree.

In the *Grundlagen,* Frege argued that according to this criterion, number-words are singular terms, so that if statements about them are ever objectively true or false, they must be so in virtue of properties of the objects which are identified and individuated in assertions of numerical identities. The logicist thesis that the truths of mathematics are derivable from the truths of logic by logical means alone accordingly entails that numbers are purely logical objects, in the sense that the identities which express the recognition and individuation of these objects are themselves logical truths. Sluga's ingenious suggestion is that Frege's concern in "ÜSB" with the nature of *synthetic* or potentially knowledge-extending identities specifying ordinary objects should be understood as a stage in the working out of his mature account of analytic (logically true) identities required for the adequate specification of the logical objects treated in the *Grundgesetze.* The specific interpretive use to which Sluga puts this general insight is hard to warrant, however.

For he claims that the difference between these two sorts of identities resides in the fact that the identities by which logical objects are identified and individuated express coincidence not just of reference but also of sense.[48] It is not clear what reasons there are to accept this reading, nor what interpretive advantages would accrue from doing so. For Frege explicitly affirms on a number of occasions that the two expressions '2^2' and '2 + 2' express different senses. And he seems committed to this view by structural principles of his approach, in particular by the compositionality principle as it applies to senses. Different function-expressions appear in these two complex designations, and the senses of components are parts of the senses of complexes containing them. Nor does the fact that such identities are to be logically true entail that they express identities of sense rather than merely of reference. Identity of sense would of course be sufficient for identity of reference. But we are often told that logic need be concerned only with truth and reference, and Frege's view seems to be that it can be logically true that two different senses determine the same reference.

This mistake aside, Sluga's tracing of the development of Frege's attempts to define abstract objects of the sort instantiated by logical objects is a valuable contribution, and raises issues of the first importance for our understanding of the constraints on interpretations of Frege's technical concepts. The story begins with the second definition of number which Frege tries out in the *Grundlagen (GL).* It states that two concepts have the same number associated with them if and only if the objects those concepts are true of can be correlated one-to-one.[49] He rejects such a definition as inadequate to specify numbers as objects, on the grounds that it will not determine whether, for example, Julius Caesar or England is identical to any number. Such a definition settles the truth values of identities (and hence the appropriateness of substitutions) only for terms which are the values for some argument expression of the function-expression "the number of the concept . . ." This procedure would be legitimate only if we had independently defined the concept (function from terms to truth values) number signified by this function-expression. But it is not possible simultaneously to specify that function and the objects for which it yields the value True. If objects had been specified by this definition, then there would be a fact of the matter as to whether Julius Caesar was one of them. But the definition does not settle this issue either way. On the basis of this objection, Frege motivates his third and final definition of numbers, considered below.

Sluga traces through the later works Frege's efforts to clarify the specification of numbers in such a way that it will not be subject to this objection, culminating in the *Grundgesetze (GG)* account of courses of values. Given the centrality to Frege's project of producing an adequate definition of number, this progress is of interest for its own sake. But the task of responding to the objection to the second *GL* definition of number is made especially urgent for interpreters of Frege by a consideration which Sluga does not mention. For the specifications of the abstract objects in terms of which Frege's semantic analysis proceeds (e.g., sense, reference, thought, truth value) are of the same objectionable form as the second *GL* definition of number. Nothing we are ever told about the senses of singular terms or sentences, for instance, settles the question of whether Julius Caesar can be such a sense. Though this may seem like a question of no interest, some interesting questions do take this form. For in interpreting the notion of sense, one is concerned *both* with subdividing the explanatory functional role played by the concept (as exhibited in the discussion of Bell) *and* with the possibility of identifying senses with things otherwise described—for example, the uses of expressions, sets of possible worlds, mental representations. Frege himself addresses such issues when he denies that the senses of sentences are to be identified with ideas in people's minds. How is the identity he wishes to deny given a sense?

All that is given is a criterion determining when the senses associated with two expressions are the same (namely, if they are intersubstitutable without change of cognitive value—*Erkenntniswerte*). If something is not specified as the sense associated with an expression (compare: number associated with a concept), its identity or nonidentity with anything which has been so given is entirely undetermined. Frege's procedure for introducing his technical concepts such as sense is invariably to attempt to specify *simultaneously* a realm of abstract semantic interpretants *and* a function which assigns a member of this realm to each expression.

We are, for instance, to associate truth values with sentences. But we are told only that the truth value associated with *p* is the same as the truth value associated with *q* just in case for no occurrence of *p* (either as a freestanding sentence or as a component in a more complex sentence) can a good inference be turned into a bad one by substituting *q* for that occurrence of *p* (reading the principle that good inferences never take true premises into conclusions that are not true as defining truth values in terms of the goodness of inferences). Even conjoining such a specification with the stipulation that the truth value associated with the sentence '2 + 2 = 4' is to be called "the True" does not settle the question of whether Julius Caesar is that truth value. He had better not be, for if the logicist program of *GG* is to be successful, truth values must be definable as purely logical objects. The current question is how the identity which is denied here is given a sense so that something could count as justifying that denial. The functions which associate the various kinds of semantic significances with expressions are always of the form: $f(x) = f(y)$ iff $R(x, y)$, where x and y range over expressions, and R is some relation defined in terms of the inferential potentials of those expressions. These are exactly the kind of definition Frege found wanting in *GL*.

Seeing how Frege believes he can overcome the objectionable indeterminateness of concepts such as that determined by the second *GL* definition of number is thus a matter of considerable importance for the appraisal of his success in specifying his own technical concepts, as well as for the narrower project of introducing numbers as logical objects. The third and final definition of number which Frege offers in *GL* is: "The Number which belongs to the concept *F* is the extension of the concept 'equal *(Gleichzahlig)* to the concept *F*.'"[50] The number three is thus identified with the extension of the concept, for example, "can be correlated one-to-one with the dimensions of Newtonian space." This definition does not have the form Frege had objected to. It, however, essentially involves a new concept, extension, which has not previously appeared in *GL*, nor indeed anywhere else in Frege's writings. In a footnote to the definition Frege says simply, "I assume that it is known what the extension of a concept is." Sluga points out that this definitionally unsatisfactory situation is not remedied in the remainder of the book. The result is scarcely up to the standards of definition to which Frege held others and himself. The project of *GL* could not be counted a success until and unless it could be supplemented with an account of the extensions of concepts.

Six years later, in "Funktion und Begriff," Frege offers such an account. The general notion of a function is explicated, and concepts are defined as functions from objects to truth values. The extension of a concept is defined as the "course of values" *(Wertheverlauf)* of that function. This is the first appearance of the concept of a course of values. Since extensions are reduced to them, the residual definitional burden bequeathed by *GL* is put off onto this new concept. What Frege tells us here is just that the course of values associated with function *F* is the same as the course of values associated with function *G* just in case for every argument the value assigned to that argument by *F* is the same as the value assigned to it by *G*. The trouble with such a stipulation, as Sluga says, is that it has exactly the objectionably indeterminate form of the second *GL* definition of number which it is invoked to correct. Frege wants to associate with each function a new kind of object, a course of values. This domain of objects and the function which assigns one to each function are introduced simultaneously. The result is that it has not been determined whether Julius Caesar is the course of values of any function. A given course of values has been individuated only with respect to other objects specified as the courses of values associated with various functions. In sum, the courses of values in terms of which the extensions of concepts are defined suffer from exactly the defect of definition which extensions of concepts were introduced to rectify or avoid.

In the *Grundgesetze*, when courses of values are introduced, this difficulty is explicitly acknowledged and described in the same terms used to raise the original objection in *GL* (though without reference to the earlier work). Frege introduces the same principle for determining when the courses of values of two functions are identical, and then points out that such a principle cannot be taken to determine any objects until criteria of identity and individuation have been supplied with respect to objects which are *not* given as courses of values. He proposes to supplement his definition so as to satisfy this demand. His proposal is that for each object not given as a course of values it be *stipulated* to be identical to an *arbitrary* course of values, subject only to the condition that distinct objects be identified with distinct courses of values.

Frege expresses the function which assigns to each function an object which is its course of values by means of an abstraction operator binding a Greek variable. The course of value of a function *F* is written as $'\delta(F\delta)$. Axiom V of the *Grundgesetze* tells us that:

(a) $'\delta(F\delta) = '\alpha\,(G\alpha)$ iff $(\forall x)\,[Fx <=> Gx]$.

Frege recognizes that this principle alone does not suffice to determine the identity of objects which are courses of values. To show this, he points out that if *X* is a function which yields distinct values if and only if it is applied to distinct arguments (what we may call an "individuation-preserving" function), then:

(a′) $X('\delta(F\delta)) = X('\alpha\,(G\,\alpha))$ iff $(\forall x)\,[Fx <=> Gx]$

without its having been settled, for instance, whether

(a″) $X('\delta(F\delta)) = '\alpha\,(G\,\alpha)$

for any *F* and *G* (including the case in which $F = G$). The by now familiar point is that (a) determines only the truth values of *homogeneous* identities, those both terms of which are of the form $'\delta(F\delta)$. And (a′) determines only the truth values of identities which are homogeneous in that both terms have the form $X('\delta(F\delta))$. But (a″) asks about *heterogeneous* identities, whose terms are of different forms. Another identity which is heterogeneous and whose truth value is accordingly not settled by principle (a) is Julius Caesar = $'\delta(F\delta)$.

To fix up this indeterminateness, which would result from taking Axiom V alone as the definition of courses of values, Frege proposes to supplement it by stipulating the truth values of the heterogeneous identities. Actually, he is required to specify the inferential behavior of course of value expressions in all contexts in which they can appear. In Frege's terminology such contexts are functions, so this requirement is equivalent to the demand that it be determined for every single-argument function-expression what value is designated by the substitution of any course of values expression in its argument place. Doing so will then determine all of the properties of the objects designated by expression of the form $'\delta(F\delta)$, for those properties just are concepts, that is, functions whose values are truth values. Among those properties are individuative properties, the facts corresponding to identity contexts involving course of values expressions. Thus the *Grundlagen* requirement that to introduce a new set of objects one must settle all identities involving them is in the *Grundgesetze* motivated by the omnicontextual condition. (It is worth noticing, as Sluga points out, that there is an endorsement of a strong context principle in Frege's claim that what it is to have introduced expressions of the form $'\delta(F\delta)$ as the names of definite objects is for the truth values of all sentential contexts in which those expressions can be substituted to have been settled.) In fact, in the spare environment of *GG* it turns out that it is not only necessary to settle the truth values of all identities involving course of value expressions in order to satisfy the omnicontextual requirement, but sufficient as well.

Indeed, in the system of the *Grundgesetze* at the time courses of values are introduced, the only objects already defined are the two truth values, and so the only heterogeneous identities Frege explicitly addresses are those involving a course of values and a truth value. But he must justify the *general* procedure of stipulating truth values for heterogeneous identities, and not just his application of it. For if he does not, then the *GG* definition of number will still be open to the objection to the second *GL* account of number (that it has not been settled whether Julius Caesar is one) which drove him to define the extensions of concepts and hence courses of values to begin with. Indeed, the concept logical object will not have been defined if it has not been settled whether Julius Caesar is one. Further, as we have seen, Frege's own definitions of his technical terms in general suffice only to determine the truth values of homogeneous identities, for example, identities of two truth values, or two senses, or two references, but not the heterogeneous identities which would be required to make the claim that Julius Caesar = the *Bedeutung* of the expression 'Julius Caesar', or that a certain linguistic role is the sense of some expression.

In particular, Frege's substitutional-inferential methodology determines only the nonrelational sense of *'Bedeutung'*, according to which expressions are sorted into substitutional equivalence classes as having the same *Bedeutung*. For Frege to add to this determination of homogeneous identities (both of whose terms are of the form "the Bedeutung of the expression *t*") the relational sense of reference in which these *Bedeutungen* are *identified* with objects suitably related to all and only the members of the nonrelational substitutional equivalence class of expressions is precisely to stipulate the truth values of the *hetero*geneous identities. The question of whether such a procedure can be justified on Frege's own terms is thus exactly the question of whether the two notions of *Bedeutung* can be made into "two aspects of *one* notion," as Dummett claims and Frege is committed to, or whether they are simply conflated without warrant, as Bell claims. Following Sluga's development of Frege's attempted definition of terms which refer to logical objects thus leads to the argument which must justify the identification of the things playing the two explanatory roles which Bell has shown must be distinguished under the heading *Bedeutung*.

In section 10 of *GG*, Frege offers his justification of the procedure of stipulating the heterogeneous identities, in an argument which Gregory Currie has called "brilliantly imaginative."[51] The argument is a difficult one, and we shall have to examine it with some care. What is to be shown is that it is legitimate to stipulate (a) above, determining the homogeneous identities involving courses of values, together with the following stipulation for heterogeneous identities:

(b) $'\tau(L\tau) = t_1$ and $'\sigma(M\sigma) = t_2$

where $t_1 \neq t_2$ and $(\exists x)(Lx \neq Mx)$. L and M are to be arbitrary functions, and t_1 and t_2 are terms which are not of the form $'\alpha\ (F\alpha)$. For the purposes of the *GG* argument, the terms in question are "the True" and "the False." In the context of Sluga's point that Frege's defense of his own view against his objection to the second attempted definition of number in *GL* must be traced through the account of extension to the account of courses of values, it will be worth keeping in mind that for this purpose the argument must apply equally to the case in which t_1 is "Julius Caesar" and t_2 is "England." To emphasize this requirement, the exposition of Frege's argument which follows will use those values for t_1 and t_2 rather than the truth values which Frege employed. In any case, the point is that distinct objects which are *not* given as courses of values are stipulated to be identical to the courses of values of a like number of arbitrary distinct functions. The task is to show that such a stipulation is legitimate.

The strategy of the argument is to *construct* a domain of objects of which (a) and (b) can be *proven* to hold. To start, suppose it has been stipulated that:

(c) $\sim\eta(F\eta) = \sim\gamma(G\gamma)$ iff $(\forall x)\ [Fx <=> Gx]$,

that is, we stipulate the homogeneous identities for terms of the form $\sim\eta(F\eta)$, where the function which associates objects so denominated with functions F is unknown except that principle (c) holds. As was pointed out above by means of (a′) and (a″), the fact that both (a) and (c) hold does not in any way settle the heterogeneous identities one of whose terms is a course of values and the other of which is of the form $\sim\eta(F\eta)$. The next step is to use the arbitrary distinct functions L and M of (b) to construct an individuation-preserving function X as above. The function X is defined by five clauses:

(1) $X(\text{Julius Caesar}) = \sim\eta(L\eta)$
(2) $X(\sim\eta(L\eta)) = \text{Julius Caesar}$
(3) $X(\text{England}) = \sim\gamma(M\gamma)$
(4) $X(\sim\gamma(M\gamma)) = \text{England}$
(5) For all other y, $X(y) = y$.

The function X is constant except when it is applied to either the two objects which are not specified as the result of applying ~-abstraction to some function (Julius Caesar and England, or the True and the False) or to the result of applying ~-abstraction to the arbitrarily chosen functions L and M. In these special cases, the function X simply permutes the distinguished values.

X is constructed to be individuation preserving, so that a correlation is preserved between distinctness of its arguments and distinctness of its values. It follows then that:

(d) $X(\sim\eta(F\eta)) = X(\sim\gamma(G\gamma))$ iff $(\forall x)[Fx <=> Gx]$.

In these terms we could now *define* the course of values notation (which has not previously appeared in this argument) by agreeing to let:

(e) $'\alpha(F\alpha) =_{df} X(\sim\eta(F\eta))$ for all functions F.

Given the definition (e) and the truth of (d), principle (a) for courses of values follows immediately. The truth of (d), as we have seen, follows from (c), together with clauses (1)–(5) defining the function X. But clauses (2) and (4) of that definition, together with (e), entail principle (b) concerning courses of values (with the substitution of Julius Caesar for t_1 and England for t_2). Thus, given only the homogeneous identities in (c), we have constructed courses of values in such a way that their homogeneous identities in (a) can be shown to hold *and* in such a way that heterogeneous identities can be *proven* for two of them, since $'\alpha(L\alpha) =$ Julius Caesar ($= X(\sim\eta(L\eta))$) and $'\delta(M\delta) =$ England ($= X(\sim\gamma(M\gamma))$). The legitimacy of stipulating heterogeneous identities in the context of a principle determining homogeneous ones has been shown by reducing the questionable stipulation to the composition of two obviously acceptable forms of stipulation: the specification of the values which the function X is to take for various arguments—in particular in clauses (2) and (4), and the introduction of the expression "$'a(Fa)$" (previously without a use) as a notational abbreviation of "$X(\sim\eta(F\eta))$."

This imaginative argument is Frege's ultimate response defending his account of number and of logical objects generally against the objections he had raised but not answered in the *Grundlagen*. Seen in that context, the argument is fallacious. The problem concerns the extremal clause (5) of the definition of the individuation-preserving function X. If that clause is expanded to make explicit what is contained in the condition "for all other y," it becomes:

(5′) $(\forall y)[(y \neq \text{Julius Caesar} \;\&\; y \neq \sim\eta(L\eta) \;\&\; y \neq \text{England} \;\&\; y \neq \sim\gamma(M\gamma)) => X(y) = y]$.

It may then be asked whether it is appropriate at this point in the argument to make use of a condition such as $y \neq \sim\gamma(M\gamma)$. If the term substituted for y is also represented as the product of applying ~-abstraction to some function, then clause (c) will settle the truth value of the resulting identity. For it settles just such homogeneous identities. But what of the case in which the identity is *heterogeneous?* All that has been fixed concerning ~-abstraction is principle (c), which says nothing about such identities. Indeed, the whole strategy of the argument depends on starting from a specification of purely homogeneous identities with one sort of abstractor (~) and using the function X to construct an abstractor (') for which the heterogeneous identities are specified. Nothing which has been said, or, given the strategy just indicated, *could* be said, settles a truth value for heterogeneous identities such as

(f) Julius Caesar $= \sim\gamma(M\gamma)$

and

(g) England $= \sim\eta(L\eta)$.

For all that principle (c) concerning ~-abstraction and the distinctness of the functions L and M settle, (f) could be true and (g) false. Given the truth of (f), substituting in clause (4) would yield that X(Julius Caesar) = England, and so by clause (1) that England $= \sim\eta(L\eta)$, that is, that

(g) is true. So the definition of X presupposes valuations for heterogeneous identities which it is in no way entitled to.

Matters are just as bad if we consider some other object, say, the direction of the Earth's axis (also discussed in *GL*). It has nowhere been determined whether it is identical to $\sim\eta(L\eta)$ and so falls under clause (2), or identical to $\sim\gamma(M\gamma)$ and so falls under clause (4), or to neither and hence falls under clause (5). The definition of X, in terms of which it is to be shown acceptable to stipulate heterogeneous identities for ~-abstraction, is well formed only if the heterogeneous identities involving ~-abstraction have already been settled. They have not been settled. Further, to add to the argument the assumption that truth values for such heterogeneous identities involving expressions of the form $\sim\delta(F\delta)$ *have* been settled is to assume exactly what the argument as a whole is supposed to show, namely, that such matters are open for stipulation in the first place (so long as suitable care is taken to match distinct objects with the result of abstracting distinct functions). If more is supposed about ~-abstraction than principle (c) fixing homogeneous identities, the question will be begged. And without some supposition about heterogeneous identities, the argument does not go through.

The intent of the offending extremal clause is to deal with all objects which can be represented by expressions of the form $\sim\delta(F\delta)$, where $F \neq L$ and $F \neq M$. Distinct objects not so representable are each to be dealt with by a pair of clauses, letting the function X permute them with the result of abstracting from corresponding arbitrarily chosen distinct functions. There is nothing in general wrong with such a definitional strategy. It may not be used in the context of this argument, however. The distinction between the cases which are to be dealt with by paired specific stipulations and those which remain to be dealt with by the extremal stipulation cannot be made precise without begging the question. For that distinction corresponds to the distinction between heterogeneous identities and homogeneous ones, in the sense of stipulations for objects not representable by expressions of the form $\sim\delta(F\delta)$ and those which are so representable. This distinction is not one which a principle like (c) specifying the purely homogeneous identities permits us to make, and we are entitled to presuppose no more than such a principle. Put otherwise, the form of definition essentially requires that there be a pair of specific clauses dealing with *every* object whose individuation with respect to the results of applying ~-abstraction to functions has not been settled by principle (c). But this class of objects cannot be described or specified in the terms permitted for the definition if it is to play its appointed role in the larger argument.

The only way in which this situation might be remedied would be if there were some property available which could be independently appealed to in order to distinguish the two kinds of cases. Thus if to (c) were added:

(c′) $(\forall y)[P(y) <=> (\exists F)(y = \sim\delta(F\delta))]$

then the extremal clause in the definition of X could be amended to

(5″) $(\forall y)[(P(y) \,\&\, y \neq \sim\eta(L\eta) \,\&\, y \neq \sim\gamma(M\gamma)) <=> X(y) = y]$

In the context of (c′), (5″) will have the desired effect of applying only to objects which can be designated by expressions of the form $\sim\delta(F\delta)$, where $F \neq L$ and $F \neq M$. More important, (c′) would ensure that the identities in (5″) are homogeneous with respect to ~-abstraction, and hence have had their truth values settled by (c). It was the failure to ensure the homogeneity that was responsible for the inadequacy of the original definition of X.

The trouble with this way out is that no such independently specifiable property is available. Already in the *Grundlagen*, Frege had pointed out that the account of when the numbers associated with two concepts were identical (settling identities homogeneous with respect to the form: the number of the concept F) could be defended against his objection if the concept <u>. . . is a number</u> were available. For then the truth values of the heterogeneous identities (such as those involving Julius Caesar) could be settled by specifying that for any t, *if* t is not a number, *then* it is not identical to the number of any concept. But the problem the desired definition was to solve was precisely that of specifying the concept <u>. . . is a number,</u> as the current task is to specify the concept <u>. . . is a course of values.</u> It would be circular to use for the property P **. . . is an x such that there is an F such that $x = \sim\delta(F\delta)$.** For that would precisely presuppose that the heterogeneous identities have somehow already been settled, rather than independently settling them. Nor could some property such as . . . **is not in the causal order** be used, for there are other logical objects (such as the True and the False) whose individuation with respect to objects specified by ~-abstraction has not been determined. Nor in any case would such a property be available to a logicist.

Frege's argument does not work, then, and it cannot be made to work. If the *Grundgesetze* is meant to offer an account of number which will meet the demands set by the *Grundlagen,* then it is a failure by Frege's own standards. Further, this failure is not a matter of the inconsistency of the later system. Although Axiom V is the culprit in both cases, it is different features of that principle which are found objectionable in the two cases. The current complaint is that settling the truth values of the homogeneous identities alone, as that principle does, is definitionally too weak to meet the requirements imposed by the discussion of *GL.* Those demand the justification of the stipulative extension of the definition to heterogeneous identities. That it leads to inconsistency, however, shows that that axiom is inferentially too strong. Putting aside the question of inconsistency which makes the claim counterfactual, even if the account of courses of values in *GG* were technically adequate, it would not be philosophically adequate as a specification of its objects and concepts. For it has not settled whether Julius Caesar is the number three, nor shown that stipulating an answer in the case of logical objects such as the truth values is a legitimate procedure. Nor can this be shown with the materials at hand.

I take it that this definitional inadequacy has not been remarked on for two connected reasons. In the purely technical context of the *Grundgesetze,* the stipulation of the two heterogeneous identities concerning the truth values and arbitrary distinct courses of values is in fact perfectly acceptable. Further, provided that it is stipulated that neither of the truth values is identical to the result of applying ~-abstraction to any function, Frege's argument shows that his procedure is in order. It is only in the larger philosophical context provided by Sluga's historical tracing of the stages in Frege's development of an answer to his own objections to the second attempted definition of number in *GL,* from the invocation of the extension of a concept in the third and final *GL* definition, via the reduction of concepts to a special kind of function and of extensions to courses of values in "Funktion und Begriff," to the final attempt to define courses of values adequately in the early sections of the *Grundgesetze* that it can be seen that satisfying the purely technical constraints will not suffice to render the definition of courses of values (and hence of logical objects generally) adequate by the philosophical standards Frege has insisted upon.

But the result is significant not just for the appraisal of the success in its own terms of Frege's account of the logical objects which were his explicit subject matter in *GG.* For as we have seen, the technical philosophical concepts Frege developed to use in that discussion, such as reference, and sense, and truth value, are all given the same form of definition as courses of values are, which individuates them only homogeneously. Thus, "we cannot say *what* the sense of an expression is. The closest we may approach to this is to say that the sense of a given expression E_1 *is the same as the sense of another expression,* E_2."[52] It follows that so far as *interpretation* (rather than further development) of Frege's concept of sense is concerned, one can *only* subdivide the explanatory roles played by his concept, but cannot identify anything as playing those roles. Thus it is legitimate and valuable to distinguish the cognitive role from the semantic role played by senses, or sense as content from sense as character, or input and output senses as Bell does. But to entertain hypotheses about whether thoughts are mental pictures (as Frege did by denying this) or sets of possible worlds, or denizens of some extracausal realm is to consider claims which have been given no sense by Frege's purely homogeneous specification of the entities in question. Truth values are similarly immune from heterogeneous identification, from identification in any other form than as the truth value associated with some expression.

Probably most important is the case of singular term reference. Here Frege tried explicitly to supplement the purely homogeneous sorting into semantic equivalence classes of the reference associated with various expressions (the nonrelational sense of '*Bedeutung*') with the stipulation of heterogeneous identities involving the references of expressions and ordinary objects. In accord with his inferential/substitutional methodology, these stipulations are grounded in the intersubstitutability for all terms *t* of the term itself and the expression 'the *Bedeutung* of *t*'. Bell has shown how much of Frege's conceptual scheme depends on the assumption that such heterogeneous identities are determined (and hence a relational sense of reference applies) for other parts of speech, given only the determination of the homogeneous identities (settling a nonrelational sense of reference) which is all that is available for expressions of these other categories. Pursuing further a line of thought Sluga initiated has shown that this assumption is indeed unwarranted, and that even Frege's attempted stipulation of coincidence of relational and nonrelational senses of 'reference' in the case of singular terms has not been justified by Frege's own standards. Thus extending Sluga's argument permits better understanding of the philosophical status of Frege's technical concepts in general, and in particular of the two sides of the concept of reference which Bell, following Dummett, has so usefully distinguished.

9

The Significance of Complex Numbers for Frege's Philosophy of Mathematics

I. Logicism and Platonism

The topic announced by my title may seem perverse, since Frege never developed an account of complex numbers. Even his treatment of the reals is incomplete, and we have only recently begun to get a reasonable understanding of how it works.[1] Presumably for that reason, the secondary literature simply does not discuss how complex numbers might fit into Frege's project.[2] As I will show, we can be quite confident from what little he does say that Frege intended his logicist program to extend to complex numbers. What we do not know is how he might have gone about it. I will try to show that *however* he approached this task, he was bound to fail. This fact has profound implications, not just for his approach to arithmetic but for his whole understanding of mathematics—and indeed, for his understanding of what is required to secure reference to particular *objects* generally.

Frege is famous for his *logicism*. This is a doctrine not about mathematics generally, but only about one part of it: arithmetic, the science that studies *numbers*. Logicism is the thesis that arithmetic can be reduced to purely logical principles, by the application of logical principles alone. But Frege endorsed a very special form of logicism, what Dummett calls *platonistic* logicism. This is the thesis that numbers are purely logical *objects*. To call something a "logical object" in Frege's sense is to say that it is an object whose existence and uniqueness can be proven, and reference to which can be secured, by the application of purely logical principles.[3]

The mere reducibility of arithmetic discourse to logical discourse need not involve the further commitment to the existence of logical *ob-*

jects. The general logicist program might instead be pursued along the lines of *Principia Mathematica*, where arithmetic discourse is analyzed in terms of second- and third-order logical properties and relations. Frege, of course, also appeals to such higher-order properties and relations. But he insists in addition that numerical expressions are singular terms, and that those that occur essentially in true arithmetic statements refer to objects of a special kind. Endorsing the reducibility thesis of logicism notoriously entails shifting the boundary Kant established between the analytic and the synthetic, so as to include arithmetic in the former category. It is less often noticed that endorsing the analysis of numbers as logical objects that is distinctive of the specifically platonistic version of logicism similarly entails shifting the boundary Kant established between general and transcendental logic. For transcendental logic in Kant's sense investigates the relationship our representations have to the objects they represent. Formal logic, Kant thought, must be silent on such aspects of content. Platonistic logicism about numbers maintains on the contrary that, at least for arithmetic discourse, purely formal logic can deliver the whole of content, including reference to objects. In his *Grundlagen der Arithmetik*, Frege is pursuing the same project of transcendental logic that Kant pursues in his first *Kritik*, albeit exclusively for a kind of non-empirical discourse.

It is precisely the platonism that distinguishes Frege's variety of logicism that I will claim cannot be made to work for the case of complex numbers. Usually when questions are raised about Frege's logicism, the focus is on the claim that numbers are *logical* objects. But I will ignore those troubles and focus on the claim that they are logical *objects*. The difficulty is that structural symmetries of the field of complex numbers collide with requirements on singular referentiality that are built deep into Frege's semantics. That collision raises fundamental questions about Frege's conception of objects—and so about commitments that are at least as central as his logicism. After all, Frege eventually gave up his logicist project, in the face of Russell's paradox, while he never gave up either his platonism or the conception of objects that turns out to cause the difficulties to be identified here.

II. Singular Terms and Complex Numbers

Frege introduces what has been called the "linguistic turn" in analytic philosophy when in the *Grundlagen* he adopts the broadly Kantian strategy of treating the question of whether numbers are objects as just another way of asking whether we are entitled to introduce singular terms to pick them out. Although Frege's avowed topic is a very special class of terms and objects, namely numerical ones, it turns out that this narrow class is particularly well suited to form the basis of a more general investigation of the notions of singular term and object. For one thing, natural numbers are essentially what we use to count, and objects in general are essentially countables. So Frege's account of counting numbers depends on his discussion of the ordinary, nonmathematical, sortal concepts that individuate objects. For another, one evidently cannot hope to understand the semantic relation between singular terms and the objects they pick out simply by invoking causal relations between them (relations of empirical intuition, in Frege's neo-Kantian vocabulary) if the objects in question are, for instance, abstract objects. Since there are no causal (or intuitive) relations in the vicinity, one must think more generally about what it is for a term to pick out a particular object.[4]

Singular terms are essentially expressions that can correctly appear flanking an identity sign.[5] The significance of asserting such an identity is to license intersubstitution of the expressions flanking it, *salva veritate*.[6] If we understood how to use one paradigmatic kind of singular term, those principles would tell us how to extend that understanding to the rest. Frege takes *definite descriptions*, in which "a concept is used to define an object," as his paradigm:

> We speak of "*the* number 1," where the definite article serves to class it as an object.[7]

> The definite article purports to refer to a definite object.[8]

The question of when we are entitled to use an expression as a singular term—as "purporting to refer to a definite object," and in case the claim it occurs in is true, as succeeding in doing so—then reduces to the question of when we are justified in using the definite article.[9] The conditions Frege endorses are straightforward and familiar:

> If, however, we wished to use this concept for defining an object falling under it [by a definite description], it would, of course, be necessary first to show two distinct things:
> 1. that some object falls under the concept;
> 2. that only one object falls under it.
>
> Now since the first of these propositions, not to mention the second, is

false, it follows that the expression "the largest proper fraction" is senseless.[10]

Securing reference to particular objects (being entitled to use singular terms) requires showing *existence* and *uniqueness*. (This requirement is not special to definite descriptions, as Frege's discussion of criteria of identity and the need to settle the truth of recognition judgments shows. It is just that the definite article makes explicit the obligations that are always at least implicitly involved in the use of singular terms.)

In the context of these thoughts, Frege himself explicitly raises the issue of how we can be entitled to use singular terms to pick out complex numbers:

> It is not immaterial to the cogency of our proof whether "$a + bi$" has a sense or is nothing more than printer's ink. It will not get us anywhere simply to require that it have a sense, or to say that it is to have the sense of the sum of a and bi, when we have not previously defined what "sum" means in this case and *when we have given no justification for the use of the definite article.*[11]

> Nothing prevents us from using the concept "square root of -1"; but we are not entitled to put the definite article in front of it without more ado and take the expression "the square root of -1" as having a sense.[12]

What more is required? To show the existence and uniqueness of the referents of such expressions. Usually in discussions of Frege's logicism, questions are raised about what is required to satisfy the *existence* condition. In what follows, I ignore any difficulties there might be on that score and focus instead on the at least equally profound difficulties that arise in this case in connection with the *uniqueness* condition.

> *How are complex numbers to be given to us* then . . . ? If we turn for assistance to intuition, we import something foreign into arithmetic; but if we only define the concept of such a number by giving its characteristics, if we simply require the number to have certain properties, then there is no guarantee that anything falls under the concept and answers to our requirements, and yet it is precisely on this that proofs must be based.[13]

This is our question. The sense of "given to us" is not to begin with an *epistemic* one but a *semantic* one. The question is how we can be entitled to use singular terms to pick out complex numbers—how we can stick our labels on *them*, catch them in our semantic nets so that we can talk and think about them at all, even falsely.

III. The Argument

Here is my claim: In the case of complex numbers, one cannot satisfy the uniqueness condition for the referents of number terms (and so cannot be entitled to use such terms) because of the existence of a certain kind of symmetry (duality) in the complex plane. Frege's *semantic* requirements on singular term usage collide with basic *mathematical* properties of the complex plane. This can be demonstrated in three increasingly rigorous and general ways.

1. *Rough-and-ready* (quick and dirty): Moving from the reals to the complex numbers requires introducing the imaginary basis i. It is introduced by some definition equivalent to: i is *the* square root of -1. But one of the main points of introducing complex numbers is to see to it that polynomials have *enough* roots—which requires that *all* real numbers, negative as well as positive, have *two* square roots. In particular, once i has been properly introduced, we discover that $-i$ is *also* a square root of -1. So we can ask: *Which* square root of -1 is i? There is no way at all, based on our use of the real numbers, to pick out one or the other of these complex roots *uniquely*, so as to stick the label "i" onto it, and not its conjugate.

Now if we ask a mathematician, "*Which* square root of -1 is i?" she will say, "It doesn't matter: pick one." And from a *mathematical* point of view this is exactly right. But from the *semantic* point of view we have a right to ask how this trick is done. How is it that I *can* "pick one" if I cannot tell them apart? What must I do in order to be *picking* one—and picking *one*? For we really *cannot* tell them apart—and as the results below show, not just because of some lamentable incapacity of ours. As a medieval philosopher might have said, they are merely *numerically* distinct. Before we proceed, it is worth saying more precisely what the denial that the uniqueness condition on singular reference can be satisfied for complex numbers actually comes to.

2. *More carefully:* The extension of the reals to the complex numbers permits the construction of a particular kind of *automorphism* (indeed, it is an *involution*, a principle of duality—but our argument will not ap-

peal to the cyclic properties that distinguish this special class of automorphisms), that is, a function that:

is 1 − 1 and onto, with domain and range both being the complex numbers;
is a homomorphism with respect to (that is, that respects the structures of) the operations that define the complex plane, namely, addition and multiplication;
has a fixed basis, that is, is an identity mapping on the reals.

Such an automorphism (homomorphism taking the complexes into themselves)—call it a "fixed-basis automorphism"—is:

(i) a *trivial* (identity) mapping for the base domain of the definition (the reals), and
(ii) a *non*trivial mapping for the extended domain (the rest of the complex plane).

The existence of such a fixed-basis automorphism would show that the extended domain cannot be *uniquely* defined in terms of the basis domain—in this case, that the reals (together with the operations of complex addition and multiplication on pairs of them) do not suffice *uniquely* to identify or define particular complex numbers.

Here is such a mapping, taking each complex number into its *complex conjugate:*

$$f(x + yi) = x - yi$$

If r is real, $f(r) = r$; so the basis is fixed.
Clearly the mapping is 1 − 1 and onto.
The complex plane is an algebraic *field,* which can be represented by a set of pairs of real numbers, together with operations of addition and multiplication.
So to show that f is a homomorphism, it must be shown that:

(a) $f[(a+bi) + (c+di)] = f(a+bi) + f(c+di)$ and
(b) $f[(a+bi) * (c+di)] = f(a+bi) * f(c+di)$.

To see (a): By the definition of +,

$$(a+bi) + (c+di) = (a+c) + (b+d)i.$$

So by the definition of f,

$$f[(a+bi) + (c+di)] = f[(a+c) + (b+d)i] =$$
$$f(a+bi) = a\text{-}bi, \text{ and } f(c+di) = c\text{-}di.$$
$$(a-bi) + (c-di) = (a+c) + (-b-d)i = (a+c)-(b+d)i.$$

To see (b): By the definition of *,

$$(a+bi) * (c+di) = (ac\text{-}bd) + (ad+bc)i.$$
$$f[(ac-bd) + (ad+bc)i] = (ac-bd)-(ad+bc)i.$$
$$f(a+bi) * f(c+di) = (a-bi) * (c-di) =$$
$$(ac-(-b)(-d)) + (-ad-bc) = (ac-bd)-(ad+bc)i.$$

So f is a fixed basis automorphism with respect to +, *, which extends $\mathfrak{R}$ to C.

3. Using a bit of (well-known) *algebraic power* to establish the same result with greater generality:

Definition: Let E be an algebraic extension of a field F. Two elements, $\alpha, \beta \varepsilon$ E are *conjugate over* F if $\mathrm{irr}(\alpha, F) = \mathrm{irr}(\beta, F)$, that is, if α, β are zeros of the same irreducible polynomial over F.
Theorem: The *Conjugate Isomorphism Theorem* says: Let F be a field, and let α, β be algebraic over F with $\deg(\alpha, F) = n$. The map $\Psi_{\alpha,\beta}$: $F(\alpha) \rightarrow F(\beta)$ defined by

$$\Psi_{\alpha,\beta}(c_0 + c_1\alpha + \ldots + c_{n-1}\alpha^{n-1}) = c_0 + c_1\beta + \ldots + c_{n-1}\beta^{n-1}$$

for $c_i \, \varepsilon \, F$ is an isomorphism of $F(\alpha)$ onto $F(\beta)$ if and only if α, β are conjugate over F.
Fact: The *complex conjugates* appealed to in defining the fixed-basis automorphism f in the argument above are *conjugate over* $\mathfrak{R}$ in the sense of the previous definition and theorem. For if $a, b \, \varepsilon \, \mathfrak{R}$ and $b \neq 0$, the complex conjugate numbers $a + bi$ and $a - bi$ are both zeros of $x^2 - 2ax + a^2 + b^2$, which is irreducible in $\mathfrak{R}[x]$.

The upshot of these results is that systematically swapping each complex number for its complex conjugate leaves intact all the properties of the real numbers, all the properties of the complex numbers, and all the relations between the two sorts of numbers. It follows that those properties and relations do not provide the resources to describe or otherwise pick out complex numbers uniquely, so as to stick labels on them. So it is *in principle* impossible to satisfy Frege's own criteria for being entitled to use complex-number designators as singular terms—that is, terms that purport to refer to definite objects. Frege is *mathematically* precluded from being entitled *by his own semantic lights* to treat complex numbers as *objects* of any kind, logical or not. Platonistic logicism is false of complex numbers. Indeed, given Frege's strictures on reference to particular objects, *any* and *every* kind of platonism is false about them. (At the end of this chapter I suggest one way those strictures might be relaxed so as to permit a form of platonism in the light of these observations.)

These are the central conclusions I want to draw. The results can be sharpened by considering various responses that might be made on Frege's behalf. But first it is worth being clear about how the problem I am raising differs from other criticisms standardly made of Frege's logicist program.

IV. Other Problems

Here are some potential problems with Frege's logicism that should *not* be confused with the one identified here. First, the problem does not have to do with whether the logicist's reduction base is really *logical*. This is the objection that arithmetic is not really being given a logical foundation, because one branch of mathematics is just being reduced to another: set theory. (For to perform the reduction in question, logic must be strengthened so as to have expressive power equivalent to a relatively fancy set theory.) One of the main occupations of modern mathematics is proving representation and embedding theorems that relate one branch of mathematics to another. One gains great insights into the structures of various domains this way, but it is quite difficult to pick out a privileged subset of such enterprises that deserve to be called "foundational."

Second, the problem pointed out here does not have to do with the definition of extensions—Frege's "courses of values." All the logical objects of the *Grundgesetze* are courses of values, and various difficulties have been perceived in Frege's way of introducing these objects as correlated with functions. Of course, one feature of Axiom V of the *Grundgesetze* (where courses of values are defined) that has seemed to

some at least a minor blemish is that it leads to the inconsistency of Frege's system—as Russell pointed out. This is indeed a problem, but it has nothing to do with *our* problem. Although it is a somewhat unusual counterfactual, there is a clear sense in which we can say that the issue of how a platonistic logicist might satisfy the uniqueness condition so as to be entitled to introduce singular terms as picking out complex numbers would arise even if Frege's logic *were* consistent.

Again, the method of abstraction by which logical objects are introduced has been objected to on the grounds that it suffers from the "Julius Caesar problem" that Frege himself diagnosed in the *Grundlagen*.[14] As he puts it there, if we introduce *directions* by stipulating that the directions of two lines are identical just in case the lines are parallel, we have failed to specify whether, for instance, Julius Caesar is the direction of any line. The worry considered here does not have this shape, however; the question is not whether the logical objects that are complex numbers can be identified with anything not so specified, but rather in what sense two objects specified as complex numbers can be told apart in the case where they are related as complex conjugates of each other.

Nor is the problem whether or in what sense Frege can be successful in demonstrating the *existence* of complex numbers as logical objects. The issue concerns the uniqueness condition on entitlement to use singular terms, not the existence condition. Indeed, the concern here should be distinguished from two other sorts of objections to Frege's procedure that can be forwarded under the heading of uniqueness. In "What Numbers Could Not Be,"[15] Paul Benacerraf argues that there can be no sufficient reason to identify numbers with one set-theoretic object rather than another—for instance, no reason to identify 0,1,2,3 . . . with, for example:

$$\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\} \ldots$$

rather than with

$$\emptyset, \{\emptyset\}, \{\emptyset,\{\emptyset\}\}, \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\} \ldots$$ [16]

This is indeed a uniqueness problem, but it concerns the uniqueness of an identification of the complex numbers with things apparently of *another* kind, logical or purely set-theoretic objects specified in a different vocabulary. Our problem arises within complex-number talk itself.

Finally, the uniqueness problem for complex numbers identified here should be distinguished from the uniqueness problem that arises from the methodology of piecemeal extensions of definitions of number in the *Grundgesetze* (a methodology that Frege elsewhere rails against). Natural numbers, for instance, are initially defined as in the *Grundlagen*. But then rational numbers are defined as ordered pairs of integers. Since the natural numbers *are* (also) rational numbers, this raises a problem: What is the relation between, say, the rational number <2,4> and the natural number 2? Will the true natural number please stand up? This uniqueness problem ramifies when the reals are defined (or would if Frege had finished doing so), since both natural numbers and rational numbers are also real numbers. Frege does not say how he would resolve this problem.

V. Possible Responses

With the problem of how one might satisfy the uniqueness requirement on the introduction of singular terms for the case of designations of complex numbers identified and distinguished from other problems in the vicinity, we can turn to possible responses on Frege's behalf. In this section we consider four ultimately inadequate responses. In the following section we consider a more promising one.

One response one might entertain is "So much the worse for the complex plane!" Or, to paraphrase Frege when he was confronted with the Russell paradox: "(Complex) arithmetic totters!" That is, we might take ourselves to have identified a hitherto unknown surd at the basis of complex analysis. Even though this branch of mathematics seemed to have been going along swimmingly, it turns out on further reflection, we might conclude, to have been based on a mistake, or at least an oversight. But this would be a ridiculous response. The complex plane is as well studied and well behaved a mathematical object as there is. Even when confronted with the inconsistency of the only logic in terms of which he could see how to understand the natural numbers, Frege never seriously considered that the problem might be with *arithmetic* rather than with his account of it. And if principles of semantic theory collide with well-established mathematical practice, it seems clear that we should look to the former to find the fault. So, confronted with the difficulty we have identified, Frege never *would* have taken this line, and we *should* not take it.

A second response might be exegetical: perhaps Frege did not intend

his logicist thesis to extend to complex numbers. After all, he only ever actually got as far as taking on the reals. Or, to vary the response, even if he was at one time a logicist about complex numbers, perhaps that is something he changed his mind about. Neither of these suggestions can be sustained, however. I have already cited some of Frege's remarks about complex numbers in the 1884 *Grundlagen*. Here is another passage that makes it clear that, at least at that point, Frege intended his logicism to encompass complex numbers:

> What is commonly called the geometrical representation of complex numbers has at least this advantage . . . that in it 1 and *i* do not appear as wholly unconnected and different in kind: the segment taken to represent *i* stands in a regular relation to the segment which represents 1 . . . A complex number, on this interpretation, shows how the segment taken as its representation is reached, starting from a given segment (the unit segment), by means of operations of multiplication, division, and rotation. [For simplicity I neglect incommensurables here.] However, even this account seems to make every theorem whose proof has to be based on the existence of a complex number dependent on geometrical intuition and so synthetic.[17]

Perhaps Frege gave up this view, then? In the second sentence of the introduction to the *Grundgesetze* of 1893, Frege says:

> It will be seen that negative, fractional, irrational, and complex numbers have still been left out of the account, as have addition, multiplication, and so on. Even the propositions concerning [natural] numbers are still not present with the completeness originally planned . . . External circumstances have caused me to reserve this, as well as the treatment of other numbers and of arithmetical operations, for a later installment whose appearance will depend upon the reception accorded this first volume.

A few years after the publication of the second volume of the *Grundgesetze*, Frege writes to Giuseppe Peano:

> Now as far as the arithmetical signs for addition, multiplication, etc. are concerned, I believe we shall have to take the domain of common complex numbers as our basis; for after including these complex numbers we reach the natural end of the domain of numbers.[18]

And as we know, even when, at the end of his life, Frege gave up his logicist program to turn to geometry as the foundation of arithmetic, his plan was to identify first the complex numbers, and the rest only as special cases of these.

Since this exegetical response will not work, one might decide to ignore what Frege *actually* intended, and insist instead that what he *ought* to have maintained is that, appearances to the contrary notwithstanding, complex numbers are not really numbers. That is, they belong on the *intuitive*, rather than the *logical*, side, of Frege's neo-Kantian partition of mathematics into geometry (which calls upon pure intuition for access to its objects), and arithmetic (which depends only on pure logic for access to its objects). After all, as Frege reminds us in the passage about the geometrical interpretation of complex numbers quoted above, multiplication by the imaginary basis i and its complex conjugate $-i$ correspond to counterclockwise and clockwise rotations, respectively. According to this proposed friendly amendment, Frege's Platonist logicism is not threatened by the impossibility of satisfying the uniqueness condition for introducing terms referring to complex numbers. For that result shows only that the boundaries to which that thesis applies must be contracted to exclude the offending case.

There are two difficulties with this response. First, uniquely specifying one of the directions of rotation (so as to get the label "i" to stick to *it)* requires more than pure geometrical intuition; it requires actual empirical intuition of the sort exercised in the use of public demonstratives. Second, if it *were* possible to pick one of the directions of rotation out uniquely in pure intuition, Frege is committed to taking the distinction that would thereby be introduced not to be an objective one—and so not one on which a branch of mathematics could be based.

For the first point: That multiplication by i or $-i$ corresponds geometrically to a rotation of $\pi/2$ radians is not conventional. But which *direction* each corresponds to is entirely conventional; if we drew the axes with the positive y axis below the x axis, i would correspond to clockwise instead of counterclockwise rotation. The question then is what is required to specify one of these directions uniquely, so as to be able to set up a definite convention. This problem is the same problem (in a mathematically strong sense, which we can cash out in terms of rotations) as asking, in a world that contains only the two hands Kant talks about in his *Prolegomena*, how we could pick out, say, the *left* one—for that is the one that, when seen from the palm side, requires *clockwise* rotation to move the thumb through the position of the forefinger to the position of the little finger. In a possible world containing only these two hands, we are faced with a symmetry—a duality defined by an involution—exactly parallel to that which we confronted in the case of the complex numbers. In fact it is exactly the *same* symmetry. Manifesting it geometrically does not significantly alter the predicament. If the world in question also contained a properly functioning clock, we could pick out the left hand as the one whose thumb-to-forefinger-to-little-finger rotation went *that* way—the same way *that* clock hand moves. But demonstrative appeal to such a clock takes us outside the hands, and outside geometry.

Inside the hands, we might think to appeal to biology. Because the four bonds of the carbon atom point to the vertices of a tetrahedron, organic molecules can come in left- and right-handed versions: enantiomers. Two molecules alike in all their physical and ordinary chemical properties might differ in that, treating a long chain of carbons as the "wrist," rotation of the terminal carbon that moved from an OH group through an NH_2 group to a single H is clockwise in the one and counterclockwise in the other. The sugars in our body are all right-handed in this sense (dextrose, not levose, which is indigestible by our other right-handed components). So we might think to appeal these "internal clockfaces" in the molecules making up the hands—appealing to biology rather than to geometry. But there is nothing biologically impossible about enantiomeric Doppelgänger, and for all Kant or we have said, the hands in question could be such. To pick out the left hand, it would have to be settled how the rotations defined by *their* sugars relate to *our* clocks. And biology won't settle *that*.

Similarly, we cannot break the symmetry of chirality, of handedness, by appeal to physics. The right-hand screw rule is fundamental in electromagnetic theory: If current flows through a wire in the direction pointed to by the thumb, the induced magnetic field spirals around the wire in the direction the fingers curl on a right hand: counterclockwise. But this fact does not give us a nondemonstrative way to specify counterclockwise rotation. For antimatter exhibits complementary chiral behavior. There is nothing physically impossible about antimatter hands, and for all Kant or we have said, the hands in question could be such. To pick out the left hand, it would have to be settled how the rotations defined by *their* charged particles relate to *our* clocks. And physics will not settle that.

So the geometrical interpretation in terms of directions of rotation will not allow us to specify uniquely *which* square root of -1 i is to be identified with, because we can only uniquely specify one direction of rotation by comparison with a fixed reference rotation, and geometry does not supply that—indeed, neither do descriptive (= nondemonstrative) biology, chemistry, or physics. This observation puts us in a position to appreciate the second point above. Even if pure geometrical intuition *did* permit us each to indicate, as it were internally, a reference direction of rotation ("By *i* I will mean *that* [demonstrative in pure inner intuition] direction of rotation"), nothing could settle that you and I picked the *same* direction, and so referred to the *same* complex number by our use of *i*. For the symmetry ensures that nothing we could *say* or *prove* would ever distinguish our uses. Frege considers a parallel case in the *Grundlagen:*

> What is objective . . . is what is subject to laws, what can be conceived and judged, what is expressible in words. What is purely intuitable [*das rein Anschauliche*] is not communicable. To make this clear, let us suppose two rational beings such that projective properties and relations are all they can intuit—the lying of three points on a line, of four points on a plane, and so on; and let what the one intuits as a plane appear to the other as a point, and vice versa, so that what for the one is the line joining two points for the other is the line of intersection of two planes, and so on, with the one intuition always dual to the other. In these circumstances they could understand one another quite well and would never realize the difference between their intuitions, since in projective geometry every proposition has its dual counterpart; any disagreements over points of aesthetic appreciation would not be conclusive evidence. Over all geometrical theorems they would be in complete agreement, only: interpreting the words differently in their respective intuitions. With the word 'point', for example, one would connect one intuition, and the other another. We can therefore still say that this word has for them an objective meaning, provided only that by this meaning we do not understand any of the peculiarities of their respective intuitions.[19]

Of course, in our case the "peculiarities of their respective intuitions" include just which complex number they indicate by '*i*'. So relinquishing logicism for the complex numbers in favor of the geometrical interpretation will not suffice to make a safe place for complex numbers in Frege's philosophy of mathematics.

As a fourth possible response, then, one might suggest that Frege give up his partition of mathematics into arithmetic and geometry: the

bits where expression and demonstration can proceed by purely logical means and the bits where pure intuition is also required. In fact, Frege never seems to have considered relinquishing this neo-Kantian demarcation. As already remarked, even when he finally despaired of founding arithmetic on logic, he turned to geometry. But in fact there is no succor available for him through such a move in any case. For the problem lies not in the conception of logic or of geometry, but in the incapacity of his semantic requirements on singular terms to accommodate certain kinds of global symmetries. But structural symmetries of the sort rehearsed in detail for the complex numbers—symmetries that preclude demonstrations of uniqueness of the sort Frege demands to secure reference to objects—are ubiquitous in modern mathematics. Here are two examples chosen almost at random:

(a) The multiplicative group U_3 of the three solutions to $x^3 = 1$, namely,

$$\{1, -1/2 + (\sqrt{3}/2)*i, -1/2-(\sqrt{3}/2)*i\}.$$

This is a concrete instance of the abstract group whose table is:

*	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

This has a permuting automorphism Ψ defined by: $\Psi(e) = e$, $\Psi(a) = b$, $\Psi(b) = a$. Similar results obtain for the abstract groups instantiated by the rest of the U_n.

(b) Klein's Viergruppe, V (which has nothing to do with complex numbers), has group table:

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

V has a permuting automorphism Ψ defined by:

$$\Psi(e) = e, \Psi(a) = c, \Psi(b) = b, \Psi(c) = a.$$

I have chosen examples from abstract group theory in part because Frege was certainly familiar with it. The definitive nineteenth-century German work on abstract algebra, Heinrich Weber's *Lehrbuch der Algebra,* was published in two volumes, the first appearing before Frege published the first volume of his *Grundgesetze,* the second well before the publication of Frege's second volume, at a time when Frege was still an active member of a mathematics department. Although Frege seems never to have used the word *'Gruppe',* in the second volume of the *Grundgesetze* he in fact proved an important theorem in group theory—one that would elude more conventional algebraists for more than fifteen years.[20]

VI. Categorically and Hypothetically Specifiable Objects

So complex numbers are just the tip of the iceberg. Large, important stretches of mathematics exhibit symmetries that preclude the satisfaction of Frege's uniqueness requirement on the introduction of singular terms. Is there any way to relax that requirement while remaining true to his motivations in introducing it? Here is a candidate. Frege's uniqueness requirement can be decomposed into two components, which we might designate *distinguishability* and *isolability.* Elements of a domain are distinguishable in case they are *hypothetically specifiable,* that is, specifiable (uniquely) *relative* to some other elements of the same domain, or *assuming* the others have already been picked out. Elements of a domain are isolable in case they are *categorically specifiable,* that is, can be specified uniquely by the distinctive role they play within the domain, or in terms of their distinctive relation to what is *outside* the domain, to what can be specified *antecedently* to the domain in question. Both of these notions can be defined substitutionally. Here are three examples: Suppose a geometer says, "Consider a scalene triangle. Label its sides 'A,' 'B,' and 'C.'" Now if someone asks, "*Which* side is to be labeled 'A'?" answers are readily available, for instance: "The one that subtends the largest angle." The case would be different if the geometer had said instead, "Consider an equilateral triangle. Label its sides 'A,' 'B,' and 'C.'" Now if someone asks "*Which* side is to be labelled 'A'?" there need be no answers available. In both cases the three sides are *distinguishable.* That is, it has been settled that the three sides are *different* from one another. For if, say, "A" and "B" labeled the *same* line segment, there would be no

triangle to discuss. So "A" could not be substituted for "B" indiscriminately, while preserving truth. And assuming that references have been fixed for "A" and "B," we can say, "'C' is the *other* side of the triangle," even in the equilateral case. But the symmetries involved in the equilateral case preclude our doing there what we can easily do in the scalene case, namely, *isolate* what the labels pick out: *categorically* specify which sides are in question.

Next, consider extending the field of the natural numbers (with addition and multiplication) to the integers. Now consider the mapping on the extension field defined by $f(n) = -n$. We could say that this mapping mapped each integer onto its *sign conjugate* (or complement). Such sign conjugates are clearly *distinguishable* from one another, for we cannot substitute "$-n$" for "n" in the second place of $n * n = n^2$, *salva veritate,* since $n*(-n) = -n^2$. Nonetheless, f is a homomorphism with respect to addition. Are the elements of the extension field nonetheless categorically specifiable? Yes. For f is *not* a homomorphism with respect to multiplication. There is an underlying asymmetry between the positive and negative integers with respect to multiplication: multiplying two positive numbers always results in a positive number, while multiplying their negative conjugates results in the same, positive number. So the positive numbers can be not only *distinguished* from the negatives (as above), but also *categorically specified* as the numbers whose sign is not changed by multiplying them by themselves.

Contrast the *complex conjugates,* which are distinguishable but *not* isolable—hypothetically but not categorically specifiable. The first notion can be defined substitutionally by looking at *local* or *piecemeal* substitutions:

$$a + bi \neq a - bi,$$

since the former cannot be substituted for the latter, *salva veritate,* in:

$$(a + bi)*(a - bi) = a^2 + b^2, \text{ while}$$
$$(a + bi)*(a + bi) = a^2 - b^2 + 2abi.$$

In this sense, the complex conjugates are *distinguishable* from one another. This means each element is *hypothetically specifiable:* specifiable *if* some other elements are.

The second demands the absence of *global* automorphisms (substitutional permutations). And that we have seen is *not* the case for the complex numbers.

Here is a third example. The group V above admits the automorphism Ψ. So its elements are not antecedently categorically specifiable (isolable). They are distinguishable, however, for if we substitute c for a in e * a = a, we get e * a = c, which is not true. Thus a and c cannot be *identified* with one another. They are *different* elements. It is just that we cannot in advance of labeling them say which is which, since the automorphism shows that they *play the same global role* in the group.

By contrast: The (nonabelian) Dihedral Group D4 of symmetries of the square consists of the following eight permutations (with the four vertices of the square labeled 1–4), together with the operation * (corresponding to composition) defined by the table below:

$\rho_0 = (1,2,3,4) \rightarrow (1,2,3,4)$	$\mu_1 = (1,2,3,4) \rightarrow (2,1,4,3)$
$\rho_1 = (1,2,3,4) \rightarrow (2,3,4,1)$	$\mu_2 = (1,2,3,4) \rightarrow (4,3,2,1)$
$\rho_2 = (1,2,3,4) \rightarrow (3,4,1,2)$	$\delta_1 = (1,2,3,4) \rightarrow (3,2,1,4)$
$\rho_3 = (1,2,3,4) \rightarrow (4,1,2,3)$	$\delta_2 = (1,2,3,4) \rightarrow (1,4,3,2)$

(So ρ_i are *rotations,* μ_i are *mirror images,* δ_i are *diagonal flips.*)

*	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	δ_1	δ_2	μ_2	μ_1
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	δ_2	δ_1	μ_1	μ_2
μ_1	μ_1	δ_2	μ_2	δ_1	ρ_0	ρ_2	ρ_3	ρ_1
μ_2	μ_2	δ_1	μ_1	δ_2	ρ_2	ρ_0	ρ_1	ρ_3
δ_1	δ_1	μ_1	δ_2	μ_2	ρ_1	ρ_3	ρ_0	ρ_2
δ_2	δ_2	μ_2	δ_1	μ_1	ρ_3	ρ_1	ρ_2	ρ_0

This group does not have a global automorphism: each element plays a unique role, and so not only is distinguishable from the others but is categorically specifiable (isolable) as well. Yet we want to be entitled to label the elements of the abstract group V, no less than those of D_4. We want to be able to say, "Call one of the elements that behaves this way [specification of its role with respect to e and b], 'a' and the other 'c.' It doesn't matter which is which."

Frege in fact recognizes this distinction. He appeals to it in distinguishing arithmetic from geometry:

> One geometrical point, considered by itself, cannot be distinguished in any way from any other; the same applies to lines and planes. Only when several points, or lines, or planes, are included together in a single intuition, do we distinguish them . . . But with numbers it is different; each number has its own peculiarities.[21]

That is, the natural numbers are *antecedently categorically specifiable (isolable),* while geometrical objects are not (though they must still be distinguishable).

Here, then, is a suggestion. We could relax Frege's uniqueness requirement on entitlement to introduce singular terms by insisting on *distinguishability* but not on *isolability*—requiring the *hypothetical* specifiability of referents but not their *categorical* specifiability. The rationale would be that this seems in fact to be what we insist on in the case of mathematical structures that exhibit the sorts of symmetry we have considered. In the context of the *Grundlagen* project where it is introduced, uniqueness mattered originally because it was necessary for countability—where once existence has been settled, the issue of one or two or more is of the essence. But *distinguishability,* by *local* substitutions that do *not* preserve truth, is sufficient for countability. For this purpose we do not *also* have to insist, as Frege does, on *categorical specifiability,* which requires the absence of certain kinds of *global* truth-preserving substitutions or permutations. Since the latter requirement would oblige us to condemn vast stretches of otherwise unimpeachable mathematical language as unintelligible or ill formed, it seems prudent to refrain from insisting on it.

There are two ways in which such a relaxation of half of Frege's uniqueness condition might be understood—confrontational or accommodating. One would construe the move as reflecting disagreement about the proper characterization of a common category of expressions: singular terms. The other would take the suggestion as recommending recognition of a second, related category of expressions: (say) schmingular terms. According to the first sort of line, Frege was just wrong in thinking that categorical specifiability is a necessary condition for introducing well-behaved singular terms. According to the second, he was quite right about one kind of singular term, what we might call "specifying" terms, and wrong only in not acknowledging the existence of another kind, what we might call "merely distinguishing" terms.

The accommodating reading is surely more attractive. The confrontational stance seems to require commitment to a substantive and (so) potentially controversial *semantic axiom of choice* that stipulates that one can label arbitrary distinguishable objects.[22] One would then naturally want to inquire into the warrant for such a postulate. Going down this road seems needlessly to multiply the possibilities for metaphysical puzzlement. Frege's practice in the *Grundlagen* would seem to show that what matters for him is that we understand the proper use of the expressions we introduce: what commitments their use entails, and how we can become entitled to those commitments. We can be entitled to use merely distinguishing terms, for instance, the labels on the sides of a hypothetical equilateral triangle, provided we are careful never to make any inferences that depend on the categorical specifiability of what is labeled—that is, that our use of the labels respects the global homomorphisms that precluded such specifiability. This is a substantive obligation that goes beyond those involved in the use of (categorically) specifying terms, so it makes sense to distinguish the two categories of singular terms. But there is nothing mysterious about the rules governing either sort. If Frege thought there was something conceptually or semantically incoherent about merely distinguishing terms, then he was wrong—as the serviceability and indispensability of the language of complex analysis (not to mention abstract algebra) shows.

VII. Conclusion

So here are some of the conclusions I think we can draw to articulate the significance of complex numbers for Frege's philosophy of mathematics. First, structural symmetries of the field of complex numbers entail that Frege's *Platonistic* or *objectivist* version of logicism cannot be made to work in his own terms for this area because of a collision with requirements on singular referentiality built deeply into his semantics. Second, as a consequence, Frege's partition of mathematics into:

(a) the study of *logical* objects, and
(b) the study of the deliverances of pure (geometrical) *intuition*

cannot be sustained in his terms. For once we have seen how things are with the complex plane, it becomes obvious that vast stretches of modern mathematics, including most of abstract algebra, will not fit

into Frege's botanization. For the sorts of global symmetries they share with the complex plane preclude Frege from allowing them in the first category, and they are not plausibly assimilated to the second. More constructively, however, I have suggested that we can make sense of reference to mathematical objects in the face of such symmetries if we are willing to relax Frege's requirements on entitlement to use singular terms, by insisting on *distinguishability* (hypothetical specifiability), but not on *categorical specifiability*.[23] Thus, looking hard at how complex numbers fit into Frege's theorizing in the philosophy of mathematics promises to teach us important lessons about the semantics of singular terms. This suggests a final general lesson: the philosophy of mathematics must pay attention to the details of the actual structures it addresses. Semanticists, metaphysicians, and ontologists interested in mathematics cannot safely confine themselves, as so many have done, to looking only at the natural numbers.