$\dot{\varepsilon}(\varepsilon^2 - \varepsilon) = \dot{\alpha}(\alpha\,.\,(\alpha - 1))$ is the value of the function $\xi = \dot{\alpha}(\alpha\,.\,(\alpha - 1))$ for the argument $\dot{\varepsilon}(\varepsilon^2 - \varepsilon)$.

§10. By presenting the combination of signs '$\dot{\varepsilon}\Phi(\varepsilon) = \dot{\alpha}\Psi(\alpha)$' as co-referential with '$\overset{\mathfrak{a}}{\smile}\,\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', we have admittedly by no means yet completely fixed the reference of a name such as '$\dot{\varepsilon}\Phi(\varepsilon)$'. We have a way always to recognise a value-range as the same if it is designated by a name such as '$\dot{\varepsilon}\Phi(\varepsilon)$', whereby it is already recognisable as a value-range. However, we cannot decide yet whether an object that is not given to us as a value-range is a value-range or which function it may belong to; nor can we decide in general whether a given value-range has a given property if we do not know that this property is connected with a property of the corresponding function. If we assume that

$$\mathrm{X}(\xi)$$

is a function that never receives the same value for different arguments, then exactly the same criterion[a] for recognition holds for the objects whose names have the form '$\mathrm{X}(\dot{\varepsilon}\Phi(\varepsilon))$' as for the objects whose signs have the form '$\dot{\varepsilon}\Phi(\varepsilon)$'. For then '$\mathrm{X}(\dot{\varepsilon}\Phi(\varepsilon)) = \mathrm{X}(\dot{\alpha}\Psi(\alpha))$' too is co-referential with '$\overset{\mathfrak{a}}{\smile}\,\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$'.[1] From this it follows that by equating the reference of '$\dot{\varepsilon}\Phi(\varepsilon) = \dot{\alpha}\Psi(\alpha)$' with that of '$\overset{\mathfrak{a}}{\smile}\,\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', the reference of a name such as '$\dot{\varepsilon}\Phi(\varepsilon)$' is by no means completely determined; at least if there is such a function $\mathrm{X}(\xi)$ whose value for a value-range as argument is not always equal to the value-range itself. Now, how is this indeterminacy resolved? By determining for every function, when introducing it, which value it receives for value-ranges as arguments, just as for all other arguments. Let us do this for the functions hitherto considered. These are the following:

$$\xi = \zeta, \ —\zeta, \ \top\zeta$$

The last one can be left out of consideration, since its argument may always be taken to be a truth value. It makes no difference whether one takes as argument an object or the value that the function $—\xi$ has for this object as argument. In addition, we can now reduce the function $—\xi$ to the function $\xi = \zeta$. For based on our stipulations the function $\xi = (\xi = \xi)$ has the same value as the function $—\xi$ for every argument; for the value of the function $\xi = \xi$ is the True for every argument. It follows from this that | the value of 17
the function $\xi = (\xi = \xi)$ is the True only for the True as argument, and that it is the False for all other arguments, just as for the function $—\xi$. After having thus reduced everything to the consideration of the function $\xi = \zeta$, we ask which values it has when a value-range appears as argument. Since so far we have only introduced the truth-values and value-ranges as objects,

[a] *Kennzeichen* — see introduction

[1] Thereby it is not said that the sense is the same.

the question can only be whether one of the truth-values might be a value-range. If that is not the case, then it is thereby also decided that the value of the function $\xi = \zeta$ is always the False when a truth-value is taken as one of its arguments and a value-range as the other. If, on the other hand, the True is at the same time the value-range of a function $\Phi(\xi)$, then it is thereby also decided what the value of the function $\xi = \zeta$ is in all cases where the True is taken as one of the arguments; and matters are similar if the False is at the same time the value-range of a certain function. Now, the question whether one of the truth-values is a value-range cannot possibly be decided on the basis of '$\dot{\varepsilon}\Phi(\varepsilon) = \dot{\alpha}\Psi(\alpha)$' having the same reference as '$\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$'. It is possible to stipulate generally that '$\tilde{\eta}\Phi(\eta) = \tilde{\alpha}\Psi(\alpha)$' is to refer to the same as '$\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', without it being possible to infer from that to the equality of $\dot{\varepsilon}\Phi(\varepsilon)$ and $\tilde{\eta}\Phi(\eta)$. We would then have, for example, a class of objects with names of the form '$\tilde{\eta}\Phi(\eta)$' for whose differentiation and recognition the same criterion would hold as for the value-ranges. We could now determine the function $\mathrm{X}(\xi)$ by saying that its value is to be the True for $\tilde{\eta}\Lambda(\eta)$ as argument, and it is to be $\tilde{\eta}\Lambda(\eta)$ for the True as argument; further, the value of the function, $\mathrm{X}(\xi)$, is to be the False for the argument $\tilde{\eta}\mathrm{M}(\eta)$, and it is to be $\tilde{\eta}\mathrm{M}(\eta)$ for the False as argument; for every other argument, the value of the function $\mathrm{X}(\xi)$[a] is to coincide with the argument itself. So, provided the functions $\Lambda(\xi)$ and $\mathrm{M}(\xi)$ do not always have the same value for the same argument, our function $\mathrm{X}(\xi)$ never has the same value for different arguments, and therefore '$\mathrm{X}(\tilde{\eta}\Phi(\eta)) = \mathrm{X}(\tilde{\alpha}\Psi(\alpha))$' is then also always co-referential with '$\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$'. The objects whose names would be of the form '$\mathrm{X}(\tilde{\eta}\Phi(\eta))$' would then also be recognised by the same means as the value-ranges, and $\mathrm{X}(\tilde{\eta}\Lambda(\eta))$ would be the True and $\mathrm{X}(\tilde{\eta}\mathrm{M}(\eta))$ would be the False. Thus, without contradicting our equating '$\dot{\varepsilon}\Phi(\varepsilon) = \dot{\varepsilon}\Psi(\varepsilon)$' with '$\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', it is always possible to determine that an arbitrary value-range be the True and another arbitrary value-range be the False. Let us therefore stipulate that $\dot{\varepsilon}(\text{—}\, \varepsilon)$ be the True and that $\dot{\varepsilon}(\varepsilon = (\top\!\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}))$ be the False. $\dot{\varepsilon}(\text{—}\, \varepsilon)$ is the value-range of the function $\text{—}\, \xi$, whose value is the True only if the argument is the True, and whose value is the False for all other arguments. All functions of which this holds have | the same value- 18
range and, according to our stipulation, this is the True. Thus $\text{—}\, \dot{\varepsilon}\Phi(\varepsilon)$ is the True only if the function $\Phi(\xi)$ is a concept under which only the True falls; in all other cases $\text{—}\, \dot{\varepsilon}\Phi(\varepsilon)$ is the False. Further, $\dot{\varepsilon}(\varepsilon = (\top\!\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}))$ is the value-range of the function, $\xi = (\top\!\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a})$, whose value is the True only if the argument is the False, and whose value is the False for all other arguments. All functions of which this holds have the same value-range and, according to our stipulation, this is the False. Every concept, therefore, under which the False and only it falls, has as its extension the

[a]Typo in Frege as noted by Thiel: Frege has '$\Phi(\xi)$' instead of '$\mathrm{X}(\xi)$'.

False.[1]

We have hereby determined the *value-ranges* as far as is possible here. Only when the further issue arises of introducing a function that is not completely reducible to the functions already known will we be able to stipulate what values it should have for value-ranges as arguments; and this can then be viewed as a determination of the value-ranges as well as of that function.

§11. Indeed we do still require such functions. If the equating of '$\acute{\varepsilon}(\Delta = \varepsilon)$' with '$\Delta$' could be maintained generally,[2] then we would have a substitute for the | definite article in language in the form '$\acute{\varepsilon}\Phi(\varepsilon)$'. For, if we assumed 19
that $\Phi(\xi)$ were a concept under which the object Δ and only this fell, then $-\!\overset{\mathfrak{a}}{\smile}\!-\,\Phi(\mathfrak{a}) = (\Delta = \mathfrak{a})$ would be the True and hence also $\acute{\varepsilon}\Phi(\varepsilon) = \acute{\varepsilon}(\Delta = \varepsilon)$ would be the True, and following our equating of '$\acute{\varepsilon}(\Delta = \varepsilon)$' and '$\Delta$', $\acute{\varepsilon}\Phi(\varepsilon)$ would be the same as Δ; i.e., in case $\Phi(\xi)$ were a concept under which one and only one object fell, '$\acute{\varepsilon}\Phi(\varepsilon)$' would designate this object. This is admittedly not possible, because the former equation had to be abandoned in its full generality; nevertheless we can help ourselves by introducing the

[1]It suggests itself to generalise our stipulation so that every object is conceived as a value-range, namely, as the extension of a concept under which it falls as the only object. A concept under which only the object Δ falls is $\Delta = \xi$. We attempt the stipulation: let $\acute{\varepsilon}(\Delta = \varepsilon)$ be the same as Δ. Such a stipulation is possible for every object that is given to us independently of value-ranges, for the same reason that we have seen for truth-values. But before we may generalise this stipulation, the question arises whether it is not in contradiction with our criterion for recognising value-ranges if we take an object for Δ which is already given to us as a value-range. It is out of the question to allow it to hold only for such objects which are not given to us as value-ranges, because the way an object is given must not be regarded as its immutable property, since the same object can be given in different ways. Thus, if we insert '$\acute{\alpha}\Phi(\alpha)$' for '$\Delta$' we obtain

$$`\acute{\varepsilon}(\acute{\alpha}\Phi(\alpha) = \varepsilon) = \acute{\alpha}\Phi(\alpha)'$$

and this would be co-referential with

$$`-\!\overset{\mathfrak{a}}{\smile}\!-\,(\acute{\alpha}\Phi(\alpha) = \mathfrak{a}) = \Phi(\mathfrak{a})',$$

which, however, only refers to the True, if $\Phi(\xi)$ is a concept under which only a single object falls, namely $\acute{\alpha}\Phi(\alpha)$. Since this is not necessary, our stipulation cannot be upheld in its generality.

The equation '$\acute{\varepsilon}(\Delta = \varepsilon) = \Delta$' with which we attempted this stipulation, is a special case of '$\acute{\varepsilon}\Omega(\varepsilon, \Delta) = \Delta$', and one can ask how the function $\Omega(\xi, \zeta)$ would have to be constituted, so that it could generally be specified that Δ be the same as $\acute{\varepsilon}\Omega(\varepsilon, \Delta)$. Then

$$\acute{\varepsilon}\Omega(\varepsilon, \acute{\alpha}\Phi(\alpha)) = \acute{\alpha}\Phi(\alpha)$$

also has to be the True, and thus also

$$-\!\overset{\mathfrak{a}}{\smile}\!-\,\Omega(\mathfrak{a}, \acute{\alpha}\Phi(\alpha)) = \Phi(\mathfrak{a}),$$

no matter what function $\Phi(\xi)$ might be. We shall later be acquainted with a function having this property in $\xi \frown \zeta$; however we shall define it with the aid of the value-range, so that it cannot be of use for us here.

[2]Compare note 1.